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ON THE POSITIVE SOLUTIONS OF THE MATUKUMA EQUATION

YI LI

1. Introduction. In 1930, based on his physical intuition, T. Matukuma proposed the following equation as a mathematical model to describe the dynamics of globular cluster of stars ([M]):

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \qquad \text{in } \mathbf{R}^3, \tag{1.1}$$

where $p > 1$ and $u > 0$ is the gravitational potential with $\int_{\mathbf{R}^3} (u^p/4\pi(1 + |x|^2))\,dx$ representing the total mass. His aim was to improve a model given earlier in 1915 by A. S. Eddington. (See [NY1,2] for a more detailed history of these two models.) Since the Matukuma equation (1.1) is rotationally invariant, the structure of positive radial solutions $u(r, \alpha)$ of the corresponding initial-value problem

$$\begin{cases} u_{rr} + \dfrac{2}{r} u_r + \dfrac{1}{1+r^2} u^p = 0 & \text{in } [0, \infty), \\ u(0) = \alpha > 0, \quad u_r(0) = 0, \end{cases} \tag{1.2}$$

was first studied by Matukuma. He then conjectured that

(i) *if $p < 3$, then $u(r, \alpha)$ has a finite zero for every $\alpha > 0$,*

(ii) *if $p = 3$, then $u(r, \alpha)$ is a positive entire solution with finite total mass for every $\alpha > 0$,*

(iii) *if $p > 3$, then $u(r, \alpha)$ is a positive entire solution with infinite total mass for every $\alpha > 0$.*

In 1938, Matukuma found an interesting exact solution $u(r; \sqrt{3}) = \sqrt{3/(1 + r^2)}$ for (1.2) with $p = 3$ which confirms part of his conjecture. Since then, there seems to have been very little mathematical contribution in the literature on this equation until the recent works of W.-M. Ni and S. Yotsutani [NY1,2], Y. Li and M.-M. Ni [LN2], and E. S. Noussair and C. A. Swanson [NS]. First, it was observed in [NY2] and [LN2] that Eddington's model *does not have any positive entire solutions* (which perhaps indicates that the Matukuma equation is indeed a better physical model). Concerning Matukuma's conjecture, [NY2] and [LN1,2] established the following results which show that equation (1.2) is perhaps more delicate than Matukuma had expected.

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