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GEOMETRIC CONSTRUCTION OF POLYLOGARITHMS

MASAKI HANAMURA AND ROBERT MACPHERSON

0. Introduction. There is currently a renaissance of research on polylogarithm functions. One branch of this activity is the effort to construct a Grassmannian p-logarithm, whose existence was conjectured in [BMS] and [HR-M]. A Grassmannian p-cocycle is a collection of analytic differential forms of various degrees on various Grassmannian manifolds which satisfies a certain cocycle condition. (The precise definition is given in §6.) A Grassmannian p-logarithm itself is one of the forms in such a collection—it is a 0-form, or a function. For an explanation of the importance of Grassmannian p-logarithms, see [BMS], [HR-M], [L, Chap. 15], [G], and [Y].

In this paper, we introduce a method of constructing analytic differential forms on algebraic varieties. We call this method *generating the forms by $\mathscr{P}$-figures.* We also develop a calculus for proving identities among differential forms generated this way, by reducing them to geometric relations among the $\mathscr{P}$-figures. Differential forms whose formulas in local coordinates are too difficult to write explicitly down can sometimes be easily constructed and manipulated with $\mathscr{P}$-figures.

We illustrate this method by using it to construct Grassmannian p-cocycles for p equal to 2 or 3, and prove the identities involved in the cocycle condition. A future paper is planned to consider the case where p is 4 or more. There are other constructions of Grassmannian polylogarithms [GM], [HR-M], [G]. However, the construction given here has some advantages. In addition to its direct use of geometry via $\mathscr{P}$-figures, it gives a direct connection to mixed Tate motives in the language of [BGSV] and [BMS].

$\mathscr{P}$-figures and the differential forms they generate. We will always denote by $\mathscr{P}$ a real Euclidean polyhedron. Fix a complex projective space $\mathbb{P}^n$. A $\mathscr{P}$-figure is an assignment of a linear subspace $M(F)$ of $\mathbb{P}^n$ to each face F of $\mathscr{P}$ (including $\mathscr{P}$ itself) such that

(i) if F is a face of $\mathscr{P}$, the complex dimension of the subspace $M(F)$ is the (real) dimension of F, and

(ii) if $F \subset F'$, then $M(F) \subset M(F')$.

For example, suppose that $\mathscr{P}$ is a square with corners v_0, v_1, v_2, v_3. Then the configuration M must be a quadrilateral of complex lines lying in a complex plane; it consists of four points $M(v_i)$, four lines $M([v_0, v_1])$, $M([v_1, v_2])$, $M([v_2, v_3])$, $M([v_3, v_0])$, and a plane $M(\mathscr{P})$ in $\mathbb{P}^3$ that are subject to the required inclusions. See Figure 0.1.

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