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LOCATING THE PEAKS OF LEAST-ENERGY SOLUTIONS TO A SEMILINEAR NEUMANN PROBLEM

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To Professor Takeshi Kotake on the occasion of his 60th birthday

1. Introduction and statement of results. In this paper we continue our study initiated in [7] and [9] on the shape of certain solutions to a semilinear Neumann problem arising in mathematical models of biological pattern formation. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and let ν be the unit outer normal to $\partial\Omega$. In [7] and [9] we considered the Neumann problem for certain semilinear elliptic equations including

$$(BVP)_d \qquad \begin{cases} d\Delta u - u + u^p = 0 \quad \text{and} \quad u > 0 \text{ in } \Omega, \\ \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $d > 0$ and $p > 1$ are constants and $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$ denotes the Laplace operator. This problem is encountered in the study of steady-state solutions to some reaction-diffusion systems in chemotaxis as well as in morphogenesis (for details, see [7] and the references therein).

Assume that p is subcritical, i.e., $1 < p < (N+2)/(N-2)$ when $N \geqslant 3$ and $1 < p < +\infty$ when $N = 2$. Then we can apply the mountain-pass lemma to obtain a least-energy solution u_d to $(BVP)_d$, by which it is meant that u_d has the smallest energy $J_d(u) = \frac{1}{2}\int_\Omega (d|\nabla u|^2 + u^2)\,dx - (p+1)^{-1}\int_\Omega u_+^{p+1}\,dx$, where $u_+ = \max\{u, 0\}$, among all the solutions to $(BVP)_d$ ([7, Theorem 2] and [9, Lemma 3.1]). It turns out that $u_d \equiv 1$ if d is sufficiently large ([7, Theorem 3]), whereas u_d exhibits a "point-condensation phenomenon" as $d \downarrow 0$. More precisely, when d is sufficiently small, u_d has only one local maximum over $\overline{\Omega}$ (thus it is the global maximum), and the maximum is achieved at exactly one point P_d on the boundary. Moreover, $u_d(x) \to 0$ as $d \downarrow 0$ for all $x \in \Omega$, while $\max u_d \geqslant 1$ for all $d > 0$ ([9, Theorems 2.1 and 2.3]).

Hence, a natural question raised immediately is to ask where on the boundary the maximum point P_d is situated, and it is the purpose of the present paper to answer this question. Indeed, we shall show that $H(P_d)$, *the mean curvature of* $\partial\Omega$ *at* P_d, *approaches the maximum of* $H(P)$ *over* $\partial\Omega$ *as* $d \downarrow 0$, as was announced in [9]. (See Theorem 1.2 below.)

Now we formulate our problem and state the results. Keeping $(BVP)_d$ in mind, first of all we formulate the problem as follows. Let Ω be a bounded domain in $\mathbb{R}^N$

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