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THE GENUS OF PROJECTIVE CURVES

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Introduction. A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree d and the dimension r of the space where they are embedded; indeed, for fixed d, it is a general fact that curves of high genus must lie in spaces of low dimension.

Halphen (1870) determined an upper bound for the genus of an irreducible curve in $\mathbb{P}^3$, and Castelnuovo (1889) found the analogous bound for the genus of irreducible, nondegenerate curves in $\mathbb{P}^r$; the sharpness of these results is shown in Castelnuovo's paper. In the permitted range, the classification of all possible genera for smooth irreducible curves is complete only in $\mathbb{P}^3$ (Gruson-Peskine), $\mathbb{P}^4$, $\mathbb{P}^5$ (Rathmann), and $\mathbb{P}^6$ (Ciliberto).

Halphen also generalized his theory to the following situation: curves C in $\mathbb{P}^3$ whose genus is big with respect to d must lie on surfaces of small degree, and so it is natural to refine the bound introducing the minimal degree s allowed for surfaces containing C. Halphen gave, in fact, a bound for the genus of space curves of degree d, not contained in surfaces of degree $< s$; his argument, however, was not complete. Gruson and Peskine ([GP]) provided a complete proof of Halphen's bound, in the range $d > s^2 - s$.

Halphen's theory can be generalized to curves in $\mathbb{P}^r$ in several ways: one may ask for the maximal genus of curves $C \subset \mathbb{P}^r$ as a function of the degree d and either

of the minimal degree allowed for hypersurfaces through C, or

of the minimal degree s allowed for *surfaces* through C.

The first point of view is wide open; for the second, results when s is not too big with respect to r are contained in [EH].

In this paper, we push further Eisenbud-Harris's point of view of [EH] and determine the bound for the genus of irreducible, nondegenerate curves in $\mathbb{P}^r$, not contained on surfaces of degree $< s$, when d is large with respect to s (i.e. $d > 2s/(r-2)\prod_{i=1}^{r-2}\sqrt[r-1-i]{(r-1)!s}$).

In our situation, it turns out that the maximal genus is the genus of some curves contained on "Castelnuovo surfaces", i.e. surfaces whose general hyperplane section is a curve of maximal genus in $\mathbb{P}^{r-1}$ (see [H2] for a description of the structure of these surfaces). The resulting value, indicated in Section 0, is proved to be an upper bound by following the classical pattern of Castelnuovo's approach (see Sections 1 and 2): First, we reduce to a general hyperplane section Z of C; by means of the differential geometric theory of "foci" on families of curves, introduced by C. Segre, we prove that Z does not lie on curves of degree $< s$ (see Section 3). Next, we

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