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AMPLE AND SPANNED VECTOR BUNDLES WITH LARGE c_1^2 RELATIVE TO c_2 ON SURFACES

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Introduction. Let E be an ample and spanned vector bundle of rank $r \geqslant 2$ on a smooth projective surface X over the complex number field $\mathbb{C}$. Here E is said to be ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$ of the projective space bundle $\mathbb{P} = \mathbb{P}_X(E)$ is ample. The problem of classifying the pair (X, E) with the small top Chern class $c_2(E)$ has been considered by several authors, in connection with some properties of the adjoint bundle $K_X \otimes \det(E)$. In fact, the pair (X, E) with $c_2(E) \leqslant 2$ are classified explicitly (see [LS], [W], [LM], [N], [LR]). So we are interested in the classification of (X, E) with the top Chern class $c_2(E) \geqslant 3$. Toward this classification, the purpose of this paper is to establish an optimal bound on $c_1(E)^2$ in terms of $c_2(E)$, by describing the pair (X, E) with large $c_1(E)^2$ relative to $c_2(E)$.

The result is the following one.

THEOREM. *Let E be an ample and spanned vector bundle of rank $r \geqslant 2$ on a smooth projective surface X over the complex number field. Then $c_1(E)^2 < (c_2(E) + 2)^2/2$ holds unless*

(1) *$c_2(E)$ is even and there exists a finite morphism $\psi: X \to \mathbb{P}^2$ of degree 2 such that $E \cong \psi^*(\mathcal{O}(1) \oplus \mathcal{O}(c_2(E)/2))$,*
(2) *$(X, E) \cong (\mathbb{P}^2, \mathcal{O}(1) \oplus \mathcal{O}(c_2(E)))$, or*
(3) *X is isomorphic to a geometrically ruled surface $\mathbb{P}_C(\mathscr{F})$ over an elliptic curve C with the projection $p: \mathbb{P}_C(\mathscr{F}) \to C$ and with the tautological line bundle $\mathcal{O}(1)$, and $E \cong p^*(\mathscr{E}) \otimes \mathcal{O}(1)$. Here $\mathscr{F}$ and $\mathscr{E}$ are indecomposable rank-2 vector bundles on C of degree 1. (Hence $c_2(E) = 2$. And E is actually ample and spanned, as proved in [BL]; see also [N].)*

Consequently, $c_1(E)^2 \leqslant (c_2(E) + 1)^2$ holds for every (X, E), and the equality holds only in case (2).

This leads to a classification of ample and spanned bundles E with $c_1(E)^2 > 4c_2(E)$ and $c_2(E) = 3, 4$.

COROLLARY. *Let E be an ample and spanned vector bundle of rank $r \geqslant 2$ on a smooth projective surface X over the complex number field. Assume that $c_1(E)^2 \geqslant 4c_2(E) + 1$.*

(1) *If $c_2(E) = 3$, then (X, E) is isomorphic to $(\mathbb{P}^2, \mathcal{O}(1) \oplus \mathcal{O}(3))$.*
(2) *If $c_2(E) = 4$, then (X, E) is one of the following:*

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