TWO TWO-DIMENSIONAL TERMINATIONS

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1. Introduction.   Varieties with log terminal and log canonical singularities are considered in the minimal model program; see [KMM] for an introduction. In [SH2] it was conjectured that many of the interesting sets associated with these varieties have something in common: they satisfy the ascending-chain condition, which means that every increasing chain of elements terminates (in [SH2] it was called the upper semidiscontinuity). Philosophically, this is one of the reasons why two main hypotheses in the minimal model program, existence and termination of flips, should be true and are possible to prove.

As for the latter, one of the main properties of flips is that log discrepancies do not decrease and some of them actually increase [SH1]. Therefore, if one could show that a set of "the minimal discrepancies" satisfies the ascending-chain condition, that would help to prove the termination of flips. The Shokurov's proof of existence of 3-fold log flips [SH3] is another example of applying the same principle. In fact, to complete the induction, it uses a 1-dimensional statement, a 2-dimensional analog of which is proved in this paper. For further discussions, see also [AK].

For one of the first examples where the phenomenon is actually proved, let us mention the following theorem.

THEOREM 1.1 ([A1], [A2]).   *Let us define the Gorenstein index of an n-dimensional Fano variety X with weak log terminal singularities as the maximal rational number r such that the anticanonical divisor $-K_X \equiv rH$ with an ample Cartier divisor H. Then a set*

$$FS_n \cap [n - 2, +\infty] = \{r(X) | X \text{ is a Fano variety and } r(X) > n - 2\}$$

satisfies the ascending-chain condition and has only the limit points $n - 2$ and $n - 2 + \frac{1}{k}, k = 1, 2, 3. \ldots$

In this paper we prove that the following two sets satisfy the ascending-chain condition:
  (i) (Theorems 3.2, 3.8) the set of minimal log discrepancies for $K_X + B$ where X is a surface with log canonical singularities and B is from a set satisfying the descending-chain condition.
  (ii) (Theorem 5.3) the set of groups $(b_1, \ldots, b_s)$ such that there is a surface X with log-canonical and numerically trivial $K_X + \sum b_j B_j$. The order on such groups is defined in a natural way; see 2.26.