Vol. 69, No. 3 DUKE MATHEMATICAL JOURNAL © March 1993

SOME GEOMETRIC ASPECTS OF GRAPHS AND THEIR EIGENFUNCTIONS

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1. Introduction. In analysis on manifolds there is an extensive literature on isoperimetric problems and the Laplacian. Some analogues in graph theory, usually concerning eigenvalues of the adjacency matrix or the associated "Laplacian," are known (see [Alo86, CDS79, Dod84]). But on the whole, much less in known for graphs, especially for isoperimetric-type problems, and many tools from analysis are in want of a good generalization to graph theory.

In this paper we show that the concept of nodal regions in analysis has a precise analogue in graph theory. This gives us geometric insight into the eigenvectors. We show how this, along with information theory and graph coverings, can give some slight improvements to certain eigenvalue bounds. We also show that the mathematical concept of a fiber product gives an interesting type of graph product; it generates new d-regular graphs from old ones in a simple manner, and numerical experiments show that it can yield graphs with small second eigenvalue. In introducing the fiber product, we discuss covering and Galois theory for graphs.

In the process of discussing the fiber product we formulate some notions of covering theory and Galois theory for graphs; while such theories are more or less known, we give a precise and concise formulation describing most situations which arise.

In Section 2 we notice that graph eigenvectors can be viewed as minimizers of a Rayleigh quotient over, say, piecewise differentiable functions on the geometric realization of the graph. This suggests a notion of "graph with boundary" and what their adjacency matrices should be. All the standard comparison theorems about eigenvalues of the Laplacian and nodal regions of eigenfunctions of the Laplacian carry over verbatim to graphs.

In particular, there is a precise graph analogue of the fact that when Dirichlet eigenfunctions of the Laplacian on manifolds are restricted to any of their nodal regions, they give the first Dirichlet eigenfunction of that region. We use this property to study some aspects of eigenfunctions and eigenvalues in what follows. We show, in Section 3, that a d-regular graph for $d \geqslant 3$ of diameter $2k$ has a second eigenvalue of at least $2\sqrt{d-1}(1 - \pi^2/(2k^2) + O(k^4))$; the proof can be stated in

Received 26 June 1992.
This paper was written while the author was on leave at the Hebrew University, Department of Computer Science. Computations were performed on both Hebrew and Princeton University facilities, including equipment purchased by a grant from DEC. The author also wishes to acknowledge the National Science Foundation for supporting this research in part under a PYI grant, CCR-8858788, and IBM for an IBM faculty development award.