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ANOTHER PROOF OF SUMIHIRO'S THEOREM ON TORUS EMBEDDINGS

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1. Introduction. In [6] Sumihiro showed that every normal algebraic variety X containing a torus $T \subset X$ in such a manner that the birational action of T on X is in fact biregular—i.e., a torus embedding—has an equivariant open cover.

Recently, Knop, et al. [3] provided a new proof. Here I provide a third proof, which is based on the results and methods of my earlier paper [2]. There I prove that, if X is isomorphic in codimension one to a torus embedding, then it is an open subset of a torus embedding. (In that paper, a torus embedding is understood to be a variety constructed from a fan. Moreover, none of the results used there depend on Sumihiro's theorem.)

In this paper we prove the following proposition.

PROPOSITION. *Suppose that $T \subset X$ is a dense open subset of a normal variety X and the birational action of T on X is biregular. Then X is isomorphic in codimension one to a variety constructed from a fan.*

From this and the theorem of [2] cited above, it follows easily that X itself is a variety constructed from a fan. It is trivial that all such varieties have an equivariant open affine cover.

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2. Proof of the Proposition. We will first prove the proposition in the special case that X contains exactly one place that does not appear on T. (A place $v = v_D$ of $k(X)$ is the rank one discrete valuation induced by a codimension one subvariety D of X.)

Take some equivariant completion $\mathbf{P}^n_{(0)}$ of T. (We suppose that $n = \dim T$.) By removing subvarieties of codimension at least 2, we can suppose that X is nonsingular [5, Ch. II, §5, Thm. 3, p111]. Again removing subvarieties of codimension at least 2, we can suppose that the birational map $\pi: X - \to \mathbf{P}^n_{(0)}$ is in fact regular [5, Ch II, §3, Thm. 3, p92]. Now, the closure $D_{(0)} = \overline{\pi(D)}$ of the image of D has dimension at most $n - 1$, and as it is T-equivariant, it cannot meet T. It must therefore lie in $\mathbf{P}^n_{(0)} - T$.

If $D_{(0)}$ has dimension $n - 1$, then we are done, for $X \to \mathbf{P}^n_{(0)}$ is a regular birational map between nonsingular varieties, which has no exceptional divisor and so is the inclusion of an open set [5, Ch II, §4, Cor. 1, p105].

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