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# COHOMOLOGY OF QUANTUM GROUPS AT ROOTS OF UNITY

VICTOR GINZBURG AND SHRAWAN KUMAR

**0. Introduction.** To any Cartan matrix $(a_{ij})$ of finite type of rank $r$, Drinfeld and Jimbo have associated a Hopf algebra $U$ over the field $\mathbb{Q}(v)$ of rational functions, with generators $E_i, F_i, K_i, K_i^{-1}$; $i = 1, \ldots, r$, subject to the quantum analogues of Serre relations (see e.g. [L2, §1.1]). Lusztig [L1], [L2] introduced a $\mathbb{Z}[v, v^{-1}]$-form $U_{\mathbb{Z}}$, analogous to the Kostant $\mathbb{Z}$-form in the classical case, as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U$ generated by the elements

$$E_i^{(n)} := E_i^n/[n]!_{d_i}, \qquad F_i^{(n)} := F_i^n/[n]!_{d_i}, \qquad K_i^{\pm 1}; \qquad n \geqslant 0, i = 1, \ldots, r.$$

(The $q$-divided powers $E_i^{(n)}$, $F_i^{(n)}$ are defined, e.g., in [L2].) Then $U_{\mathbb{Z}}$ is a Hopf subalgebra of $U$ (see [L2, §1.3]). Thus, for any $\mathbb{Z}[v, v^{-1}]$-algebra $k$, one obtains, by extension of scalars, a Hopf $k$-algebra $U_k := k \otimes_{\mathbb{Z}[v, v^{-1}]} U_{\mathbb{Z}}$.

In particular, let $\xi$ be a primitive root of unity of odd order $l$ and let $k := \mathbb{Q}(\xi)$ be the cyclotomic field, viewed as a $\mathbb{Z}[v, v^{-1}]$-algebra via the specialization $v \mapsto \xi$. The elements $K_i^l$ $(i = 1, \ldots, r)$, are then central in $U_k$, and we set

$$U_\xi := U_k/\text{Ideal generated by } \{K_i^l - 1\}.$$

Lusztig further introduced the subalgebra $u_\xi \subset U_\xi$, generated by the elements $E_i$, $F_i$, $K_i^{\pm 1}$; $i = 1, \ldots, r$, called the *restricted enveloping algebra.* He showed that $u_\xi$ is a Hopf subalgebra of $U_\xi$ of dimension $l^{\dim \mathfrak{g}}$, where $\mathfrak{g}$ is the split semisimple Lie algebra over $k$ associated to the Cartan matrix $(a_{ij})$.

Given an affine algebraic variety $V$ over $k$, let $k[V]$ denote the algebra of regular functions on $V$. If $V$ is a cone, then there is a natural grading on $k[V]$ by nonnegative integers. Let, in particular, $\mathcal{N} \subset \mathfrak{g}$ be the *nilpotent cone,* the subvariety of all the ad-nilpotent elements of $\mathfrak{g}$.

In this paper, we compute the cohomology $H^*(u_\xi, k)$ of the algebra $u_\xi$ with trivial coefficients. (See the appendix for the definition of the cohomology of an algebra.) *The following restrictions on $l$ (= order of the root of unity) are in force throughout the paper: $l$ is odd and $l > h$, and moreover $l$ is prime to 3 if* $\mathfrak{g}$ *has factors of the type $G_2$, where $h$ is the Coxeter number of* $\mathfrak{g}$ (see, e.g., [J, Page 262]).

MAIN THEOREM. $H^{\text{odd}}(u_\xi, k) = 0$, *and there is a natural graded algebra isomorphism*

$$H^{2\bullet}(u_\xi, k) \simeq k^\bullet[\mathcal{N}].$$

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