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SINGULAR MATRICES AND A UNIFORM BOUND FOR CONGRUENCE GROUPS OF $SL_n(\mathbf{Z})$

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1. Introduction. Consider a set of irreducible polynomials $\{\mathscr{F}_1, \ldots, \mathscr{F}_s\} \subset \mathbf{Z}[x_1, \ldots, x_t]$ and let

$$\mathscr{V} = \{x \in \mathbf{C}^t \colon \mathscr{F}_j(x) = 0, j = 1, \ldots, s\}.$$

It is a fundamental problem in Diophantine geometry to understand the distribution of the integral points, $\mathscr{V}(\mathbf{Z})$, on the real variety $\mathscr{V}(\mathbf{R})$ with respect to a Euclidean norm on $\mathbf{R}^t$. Specifically, one analyzes the behavior of the counting function

$$N(T, \mathscr{V}) = |\{x \in \mathscr{V}(\mathbf{Z}) \colon |x| \leqslant T\}|$$

as T grows. When these polynomials are homogeneous, one also studies the distribution of rational points on the associated projective variety. In the most general setting very little is known, and one is led to consider these problems when more structure is available.

In recent work, Duke, Rudnick, and Sarnak [DRS] study varieties that are homogeneous spaces for the action of a reductive linear algebraic group G. Assume that G acts on a vector space V and that both are defined over $\mathbf{Q}$. Consider a Zariski-closed orbit $\mathcal{O} = v_0 G$. Under certain assumptions on G and $H = \mathrm{stab}_G(v_0)$, L^2 analysis on $G(\mathbf{R})$ yields an asymptotic formula for $N(T, \mathcal{O})$, for a fixed norm on V. Specifically, assume that $c_G = \mathrm{vol}(G(\mathbf{Z})\backslash G(\mathbf{R}))$ and $c_H = \mathrm{vol}(H(\mathbf{Z})\backslash H(\mathbf{R}))$ are both finite with respect to the Haar measures dg and dh, respectively. We normalize the measures so that $c_G = c_H = 1$. Let $d\bar{g}$ be the unique G-invariant measure on $H\backslash G$ satisfying $dg = dh\,d\bar{g}$ and define

$$\mu(T) = \int_{|v_0\bar{g}| < T} 1 \, d\bar{g}.$$

Duke, Rudnick, and Sarnak prove the following statement.

THEOREM.

$$N(T, \mathcal{O}) \sim \mu(T). \tag{1}$$

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