Vol. 69, No. 1 DUKE MATHEMATICAL JOURNAL © January 1993

THE PENALIZED OBSTACLE PROBLEM. I. LIPSCHITZ REGULARITY OF LEVEL SETS

JUAN C. REDONDO

A mis padres

This is the first of two papers in which we study the uniform regularity of the level sets of solutions to the penalized obstacle problem. Here we will prove that if the solutions satisfy a density condition, then their level sets are locally graphs of Lipschitz functions. In the next paper we will show that they are, in fact, $C^{1,\alpha}$ graphs.

Consider the following obstacle problem: minimize

$$E(v) = \int_\Omega (|\nabla v(x)|^2 + 2v(x))\, dx$$

among nonnegative functions in $H^1(\Omega)$ having prescribed boundary values. The main difficulty in treating this problem lies in the fact that we restrict the set of admissible functions by asking them to be nonnegative. The method of penalization consists in putting this constraint inside the energy functional by constructing a family of functionals, $E_\varepsilon(v)$, that behave like $E(v)$, when $v \geqslant 0$, but that somehow penalize the function when it goes below zero. The smaller the ε, the greater the penalty.

As an example, consider the following penalization: minimize

$$E_\varepsilon(v) = \int_\Omega (|\nabla v(x)|^2 + F_\varepsilon(v))\, dx$$

where

$$F_\varepsilon(v) = \begin{cases} \dfrac{1}{\varepsilon} v^2 & \text{if } v \leqslant \varepsilon \\ 2v - \varepsilon & \text{if } v \geqslant \varepsilon \end{cases}$$

among functions in $H^1(\Omega)$ with prescribed boundary values. Notice that E_ε tends to discard as minimizers functions that become negative. For nonnegative functions we have that, in some sense, $E_\varepsilon \to E$. Therefore, it is natural to expect that the minimizers of E_ε will approximate the solution of the obstacle problem. That this is the case is a classical result. It was proven by H. Lewy and G. Stampacchia [LS] in the case of $W^{2,p}(\Omega)$, for any $p < \infty$, and by H. Brezis and D. Kinderlehrer [BK] for the $W^{2,\infty}(\Omega)$ norm. In both cases this was achieved by obtaining appropriate uniform a priori estimates for the solutions of the penalized problem.

Received 23 December 1991.