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CHERN CLASSES AND DEGENERATIONS OF HYPERSURFACES AND THEIR LINES

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1. Introduction. Let X be a generic hypersurface of degree d in $\mathbb{P}^n$ and $G(2, n + 1)$ be the Grassmannian of the lines in $\mathbb{P}^n$. The class of the lines contained in X is then equal to the top Chern class of the vector bundle $S^d U^*$, where U is the universal bundle on $G(2, n + 1)$. Using this one can easily compute that, for example, there are 27 lines on a generic cubic surface and 2875 lines on a generic quintic threefold. What happens when X degenerates? For example, if X becomes reducible, how can one identify the lines in each component which are the limits of the lines in a generic X and compute the class of those lines in the Chow ring of $G(2, n + 1)$? In the case of quintic threefold, Katz [K] has given a complete description for the geometry of the limiting lines. In this paper we will investigate such problems in general.

Let $\{X_s = KL + sD + \cdots\}_{s \in \Delta}$ be a degeneration of generic hypersurfaces of degree d in $\mathbb{P}^n$, where $\deg K = k$, $\deg L = l$, and $\deg D = d = k + l$. We will use the same letter to denote a hypersurface and its equation. Define

$$\sigma_K = \{\alpha \in G(2, n + 1) | \alpha \subset K, \alpha \cap L \subset D\},$$

$$\sigma_L = \{\alpha \in G(2, n + 1) | \alpha \subset L, \alpha \cap K \subset D\}.$$

Katz's results [K] suggest that one should study σ_K and σ_L in order to understand the set of limiting lines. By tensoring the section of $S^l U^*$ induced by L, we get a homomorphism ρ_L from $S^k U^*$ to $S^d U^*$ and hence have an exact sequence of coherent sheaves

$$S^k U^* \xrightarrow{\rho_L} S^d U^* \to B_K \to 0,$$

where

$$B_K = S^d U^* / \rho_L(S^k U^*)$$

is the quotient. Next, one can formally define the Chern class of B_K by the usual multiplicative formula for the quotient bundle; that is, one defines inductively

$$\begin{cases} c_0(B_K) = 1, \\ c_i(B_K) = c_i(S^d U^*) - \sum_{j=0}^{i-1} c_j(B_K) c_{i-j}(S^k U^*), \qquad i > 0. \end{cases}$$

Similarly, one defines B_L and its class. Our first result is the following theorem.

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