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RIEMANNIAN MANIFOLDS WITH CONNECTED ROYDEN HARMONIC BOUNDARIES

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To Professor Kikuji Matsumoto on his sixtieth birthday

The Royden and Wiener harmonic boundaries (see [2], [15]) play important roles in the linear potential theory, i.e., the theory of classical harmonic functions. Recently, Ishida [6] implicitly pointed out that the Royden harmonic boundary of the open unit disc in the complex plane, i.e., the open unit ball $\mathbf{B}^2$ in the Euclidean space of dimension two, is connected. The significance of this phenomenon reveals itself in the conspicuous contrast with the case of the Wiener harmonic boundary: The Wiener harmonic boundary is always totally disconnected not only for the open unit disc but also for every carrier of harmonic functions. The analytic reformulation (see 3.5 below) of the connectedness of the Royden harmonic boundary is the nonexistence of nonconstant harmonic measures with finite Dirichlet integrals which is also important in connection with the general condenser problem in the electrostatics. The Royden harmonic boundary of the unit ball $\mathbf{B}^n$ in the Euclidean space of dimension n will be seen later in this paper to be connected not only for $n = 2$ as pointed out by Ishida but also for the higher-dimensional case $n \geqq 3$ as well. This paper is motivated by the question of what happens when the Euclidean ball $\mathbf{B}^n$ is replaced by a Riemannian ball $(\mathbf{B}^n, ds)$, i.e., $\mathbf{B}^n$ with a Riemannian metric ds.

We will discuss this in a broader frame. Instead of $(\mathbf{B}^n, ds)$, we take a Riemannian manifold M of dimension $n \geqq 2$. Inheriting an important differential topological feature of the ball, M is moreover supposed to be *ball-like* in the sense that the $(n-1)$th de Rham cohomology group of M vanishes. Our problem is to examine whether the linear Royden harmonic boundary $\Delta(M)$ (see 3.1 and 3.2 below) of a ball-like Riemannian manifold M is connected or not. In the course of studying this problem, we encounter a new property of Riemannian manifolds which we call the *Virtanen property*: Any Dirichlet finite smooth $(n-2)$-form can be approximated in the Dirichlet seminorm by bounded smooth $(n-2)$-forms (see 1.4 below). The main body of this paper consists of the following two results:

(1) The Royden harmonic boundary $\Delta(M)$ of a ball-like Riemannian manifold M is connected if M possesses the Virtanen property;
(2) There exists a ball-like Riemannian manifold M with disconnected Royden harmonic boundary $\Delta(M)$ so that the Virtanen property is not satisfied by this M.

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