Vol. 67, No. 3 DUKE MATHEMATICAL JOURNAL © September 1992

THE GAUSS-GREEN THEOREM FOR FRACTAL BOUNDARIES

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1. Introduction. The Gauss-Green formula

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega, \tag{1}$$

where Ω is a compact smooth n-manifold with boundary in R^n and ω is a smooth $(n-1)$-form in R^n, is a classical part of the calculus of several variables (e.g., [Sp]).

When Ω is permitted to have positive codimension, (1) is often called Stokes's theorem; we use "Gauss-Green" to refer to the case where Ω has codimension zero. (Note that the Gauss-Green formula is often written in the equivalent form

$$\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, ds = \int_{\Omega} \operatorname{div} \mathbf{v}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, ds is the element of area on $\partial\Omega$, and $\mathbf{v}$ is the 1-vectorfield "dual" to ω: if $\omega = \sum (-1)^{i+1} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$, then $\mathbf{v} = (f_1, \ldots, f_n)$.)

There has been considerable effort in the literature (e.g., [JK], [M], [P]) to extend this formula to permit integrands of less regularity by generalizing the Lebesgue integral. On the other hand, the situations in which (1) holds require fairly strong hypotheses on the boundary $\partial\Omega$, e.g., that it should have sigmafinite $(n-1)$-measure, or that the gradient of the characteristic function of Ω be a vector-valued measure with finite total variation [F], [P].

However, there is a natural way to expand the validity of (1) to more general boundaries while still using the ordinary Lebesgue integral; this is the topic of the present paper.

For the case of Lipschitz forms, the results of this paper follow readily from Whitney's theory of flat chains [W2]. In [HN] we extend Whitney's method to treat the more general Hölder case for $n = 2$ and use (1) partly as a definition. But that paper also uses (1) partly as a definition.

A purpose of this paper is to present (1) as a theorem (see Theorem B), in which the boundary is not required to be rectifiable, and the left-hand side is defined in a

Received 12 November 1991. Received revision 13 February 1992.
First author partially supported by the National Science Foundation.
Second author partially supported by a National Science Foundation Postdoctoral Research Fellowship.