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LINES OF MINIMA IN TEICHMÜLLER SPACE

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The Teichmüller space of genus g, $g \geqslant 2$, can be defined as the space of hyperbolic metrics on a surface of genus g, up to isometry isotopic to the identity. In this paper we begin to describe a geometry of Teichmüller space T_g based on the hyperbolic geometry of the surfaces themselves, in particular, based on the length function of simple geodesics.

In the genus-1 case, T_1 is the space of flat metrics of area 1 on the torus which can be identified with 2-dimensional hyperbolic space $\mathbb{H}^2 = SL(2, \mathbb{R})/SO(2)$. The primary geometric objects in $\mathbb{H}^2$, the geodesics and horocycles, become 1-parameter families of flat structures (called deformations) under this identification. They are realized on the torus as an expansion and contraction along two geodesic foliations in the first case and as a shearing map along geodesics in the second.

There have been various attempts to generalize this geometry of T_1 to higher genus. The most successful of these is the Teichmüller geometry coming from the theory of quasi-conformal mappings and quadratic differentials. A quadratic differential determines two orthogonal measured foliations (with singularities when $g \geqslant 2$) on the surface and the geodesics in this geometry are described by an expansion and contraction respectively along these foliations. (See, e.g., [7] for a discussion.) In the toral case the leaves of these foliations are geodesics in the flat structure so that the quasi-conformal deformations are easily understood in the context of flat geometry. This is no longer true in the higher genus case; there is no apparent connection between the foliations and the hyperbolic structure on the surface and the quasi-conformal deformations translate to deformations of the hyperbolic structure only indirectly via the Riemann mapping theorem.

In this light it seems advantageous to have a geometry of T_g, $g \geqslant 2$, based directly on the hyperbolic geometry of the surfaces. We have thus chosen to look at the length function $\ell_\varphi(M)$, $M \in T_g$, of a simple closed geodesic φ generalized to the length $\ell_\mu(M)$ of a geodesic lamination μ. These length functions have been extremely useful in the study of Teichmüller space. They have been further generalized to lengths of geodesic currents by Bonahon in [1], where they form the basis for a pretty description of Teichmüller space. This leads to a new analogy between T_g and hyperbolic space, in particular, between Thurston's compactification of T_g ([11]) and that of hyperbolic space. The reader is encouraged to refer to that paper as another method of equipping Teichmüller space with geometric properties of hyperbolic space. (Also, see the end of Section 3 of this paper for a brief discussion of the relationship between that paper and the analysis here.)

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