Vol. 64, No. 3 DUKE MATHEMATICAL JOURNAL © December 1991

HEIGHTS ON THE MODULI SPACE OF RIEMANN SURFACES WITH CIRCLE BOUNDARIES

HALA HALIM KHURI

Introduction. Let $\Sigma = \Sigma_{p,n}$ be a compact surface of genus p with n distinct open disks removed. Let σ be a smooth metric on Σ which extends smoothly to $\partial\Sigma$. Associated with σ is the Laplace-Beltrami operator $\Delta = \Delta_\sigma$ with Dirichlet boundary conditions. Let $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \ldots$ denote the eigenvalues of Δ. Then $\det \Delta$, the determinant of the Laplacian, is formally $\prod \lambda_j$ and is rigorously defined by

$$\det \Delta = \exp(-Z'(0))$$

where $Z(s)$ is the zeta function

$$Z(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}.$$

This series converges absolutely for $\Re(s)$ large, and $Z(s)$ has the integral representation

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{TR}(e^{\Delta t}))t^s \frac{dt}{t}.$$

From the small time asymptotics of $(\mathrm{TR}(e^{\Delta t}))$, see [MS], one easily gets the meromorphic continuation of $Z(s)$ to the complex plane, and the zero of $\Gamma(s)^{-1}$ at $s = 0$ ensures that $Z(s)$ is regular at $s = 0$. One also has $Z(0) = \chi(\Sigma)/6$ in the case $\partial\Sigma \neq \varnothing$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ.

This definition of $\det \Delta$ was given by Ray and Singer [RS] for a compact, complex, analytic manifold without boundary, in the context of analytic torsion. More recently, $\det \Delta$ has played an important role in Polyakov string theory as part of the integrand in a Feynman-type integral over moduli space. The concern there is the set of all surfaces, with varying metrics on those surfaces, and with their associated Laplacians. In this connection Polyakov [P1], [P2] showed how to compute the change in the determinant if the metric is changed conformally and if $\partial\Sigma = \varnothing$. This was later extended to surfaces with boundary by Alvarez [A].

Set $h(\sigma) = -\log(\det \Delta)$. Then for a flat metric σ_0 on Σ and flat $\sigma = e^{2\phi}\sigma_o$, the Polyakov-Alvarez variation formula reads

Received 6 October 1990. Revision received 21 March 1991.