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DECAY OF EIGENFUNCTIONS ON SEMISIMPLE SYMMETRIC SPACES

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Introduction. Let G be a connected semisimple real Lie group with finite center, σ an involution of G, and H an open subgroup of the fixed point set of σ. The purpose of this note is to establish an analogue for eigenfunctions on the semisimple symmetric space G/H of a theorem due to R. Howe and C. C. Moore [HM]. The Howe-Moore theorem says that matrix coefficients of a unitary representation of G, not containing the trivial representation of any noncompact factor of G, must decay at infinity. In particular, let K be a maximal compact subgroup of G, and $f \in C^\infty(G/K)$ a (left) K-finite eigenfunction for the center $Z(\mathfrak{g})$ of the universal enveloping algebra of G. Suppose that the $(\mathfrak{g}, K)$-module V_f generated (on the left) by f is unitarizable, and that V_f does not contain the trivial representation of any noncompact factor of G. Then the pullback to G of f is a matrix coefficient of V_f and hence f decays at infinity on the Riemannian symmetric space G/K.

Consider now, instead of the Riemannian symmetric space, the semisimple symmetric space G/H. Let θ be a Cartan involution of G commuting with σ, K the corresponding maximal compact subgroup. Let $f \in C^\infty(G/H)$ be a (left) K-finite eigenfunction of $Z(\mathfrak{g})$. Assume as before that the $(\mathfrak{g}, K)$-module V_f generated by f is unitarizable and its restriction to any noncompact factor of G does not contain the trivial representation. We then prove (Theorem 1) that f vanishes at infinity on G/H. We also give rates of decay for f, depending on the Langlands parameters of V_f (Theorem 2).

One should note that the pullback to G of such an eigenfunction f on G/H will only be a matrix coefficient of V_f if H is compact (so that G/H is Riemannian). Indeed, this is a consequence of the Howe-Moore theorem, which mandates the vanishing of matrix coefficients in all infinite directions on G; hence, if H is noncompact, a matrix coefficient cannot be constant on right H-orbits. (However, f is a generalized matrix coefficient; see [BS, Lemma 2].) Notice also that our result actually specializes to the Howe-Moore theorem when G is considered as the symmetric space $G \times G$. We do, however, use Howe-Moore in proving the theorem.

Decay of matrix coefficients has been useful in a number of contexts such as ergodic theory [Z], representation theory [BW] and number theory, especially the theory of automorphic forms—see for example [DHL]. The generalization given in this note should have similar application; in [DRS] it is used to count the number of integral points in a ball on an algebraic $\mathbb{Q}$-variety whose real points are of the form G/H.

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