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THE POINCARÉ METRIC AND A CONFORMAL VERSION OF A THEOREM OF THURSTON

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1. Introduction. If one is to understand the quasi-conformal deformation theory of a Kleinian group, one must be able to translate information about the conformal structure at infinity into information about the internal geometry of the associated hyperbolic 3-manifold. One of the best examples of such a relationship is given by a theorem of Sullivan which says that there exists K such that, if every component of the domain of discontinuity D_Γ of a finitely generated Kleinian group Γ is simply connected, then the Poincaré metric on D_Γ/Γ is K–quasi-isometric to the hyperbolic structure on the boundary of the convex core of $\mathbb{H}^3/\Gamma$. See Epstein-Marden [9]. In this note we will show that given A there exists R such that, if Γ is any nonelementary Kleinian group such that every geodesic in the domain of discontinuity D_Γ has length at least A (in the Poincaré metric on D_Γ) and c is any closed curve in $S = D_\Gamma/\Gamma$, then

$$l_N(c^*) \leqslant R l_S(c)$$

where $l_S(c)$ denotes the length of c in the Poincaré metric on S and $l_N(c^*)$ is either equal to the length of the closed geodesic c^* in $N = \mathbb{H}^3/\Gamma$ homotopic to c or zero if no such geodesic exists. As a corollary of this observation, combined with Ahlfors's finiteness theorem, we see that, if Γ is any nonelementary finitely generated Kleinian group, then there exists some R_Γ such that $l_N(c^*) \leqslant R_\Gamma l_S(c)$ for any closed curve c on S.

We then apply our main result to study algebraic convergence of sequences of quasi-conformally conjugate Kleinian groups. In Section 3 we prove a conformal version of a compactness theorem of Thurston. In Section 4 we prove a generalization of Bers's slice theorem. In both cases we will combine Thurston's original compactness theorem, whose hypotheses are in terms of the internal geometry of the associated hyperbolic manifolds, with our main result to obtain compactness results whose conditions are in terms of the conformal structures at infinity.

2. The Poincaré metric and the internal geometry. A *Kleinian group* is a discrete faithful representation $\rho\colon G \to \mathrm{PSL}_2(\mathbb{C})$. We will denote the image $\rho(G)$ by Γ. The group $\mathrm{PSL}_2(\mathbb{C})$ is naturally identified, via the upper half-space model, with the group $\mathrm{Isom}_+(\mathbb{H}^3)$ of orientation preserving isometries of hyperbolic 3-space. The extended complex plane $\mathbb{C} \cup \{\infty\}$, regarded as the sphere at infinity for $\mathbb{H}^3$, is divided into the limit set L_Γ for Γ's action and the domain of discontinuity D_Γ. (See

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