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CONTINUATION TO GRADIENT FLOWS

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1. Introduction. In [1] Conely introduced the idea of continuation of an isolated invariant set. Roughly, if one has a parametrized family of flows and N is an isolating neighborhood for each flow in the parameter interval, then the sets isolated by N along the interval are said to be related by continuation. A fundamental result of [1] is that sets related by continuation have the same Conley index. This continuation property is especially useful for computations: one can continue a complicated invariant set to a simpler one and perform the computations in the simpler setting.

The first part of this paper is devoted to showing that any isolated invariant set on a manifold can be continued to an isolated invariant set in a gradient flow. In the nondegenerate (i.e., Morse-Smale) case, such flows are particularly simple: all bounded orbits are either critical points or orbits connecting two critical points. The Morse-Smale case can always be acheived by an arbitrarily small perturbation (i.e., continuation). In this case one can compute the $\mathbb{Z}_2$ homology of the index by counting the number of connecting orbits between points of adjacent indices. (See [2].)

If we have an attractor-repeller pair in the invariant set, then we can continue the attractor and repeller to Morse-Smale gradients. The continuation, plus counting the number of connecting orbits, gives a degree-1 map from the homology index of the original repeller to the homology index of the original attractor. We will show that this map allows us to form a chain complex whose homology is isomorphic to the homology index of the original invariant set. Thus, the continuation gives us a new approach to the flow-defined boundary map and two-set connection matrix for the Conley index theory.

The rest of this section contains background definitions and results from the theory of the Conley index. In the second section we prove that any isolated invariant set can be continued to a gradient. The third section contains background material on attractor-repeller pairs, Morse decompositions, and index filtrations, and in the fourth section we use continuation to define the boundary map for attractor-repeller pairs.

Let ϕ_t be a smooth flow on a finite-dimensional manifold M. Everything in this section is based on the ideas of Conley as described in [1]. Other references are

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