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BOUNDS FOR MULTIPLICITIES OF AUTOMORPHIC REPRESENTATIONS

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1. Let G be a semisimple Lie group of noncompact type and Γ an arithmetic lattice in G. This paper is concerned with bounding from above the multiplicity $m(\pi, \Gamma)$ with which $\pi \in \hat{G}$ occurs in the decomposition of the regular representation of G on $L^2(\Gamma\backslash G)$. By arithmetic we mean that G is defined over $\mathbb{Q}$ (or a number field $E/\mathbb{Q}$), $\Gamma \subset G(\mathbb{Q})$ and there is a $\mathbb{Q}$-embedding $\rho\colon G \to GL(n, \mathbb{R})$ such that $\rho(\Gamma)$ is commensurable with $G(\mathbb{Z}) = \rho(G) \cap GL_n(\mathbb{Z})$. Using this realization, we obtain "congruence" subgroups $\Gamma(q)$ of Γ by setting

$$\Delta(q) = \{g \in G(\mathbb{Z}) | g \equiv I(q)\} \tag{1}$$

and

$$\Gamma(q) = \rho^{-1}(\rho(\Gamma) \cap \Delta(q)). \tag{2}$$

Our main goal is to develop upper bounds for $m(\pi, \Gamma(q))$ for π fixed and $q \to \infty$. Besides the intrinsic interest in this problem, there are two interesting applications.

(1) If π contributes to cohomology theory via Matsushima's formula [B-W] (when $\Gamma\backslash G$ is compact), our upper bounds give nontrivial upper bounds for the Betti numbers of $\Gamma(q)$.

(2) By combining these bounds with representation theory of the finite group $G(\mathbb{Z})/\Delta(q)$, we can, under certain circumstances, prove that for certain $\pi \in \hat{G}$

$$m(\pi, \Gamma(q)) = 0. \tag{3}$$

That is an "arithmetic vanishing theorem". These correspond to Ramanujan like bounds on the spectrum of $\Gamma(q)\backslash G$ [S].

Fix cocompact Γ as above. The trivial upper bound is

$$m(\pi, \Gamma(q)) \ll V(q) \tag{4}$$

where $V(q) = [\Gamma, \Gamma(q)]$. The notation $A(q) \ll B(q)$ means $|A(q)| \leqslant cB(q)$ for some constant independent of q.

In fact, (4) is sharp if π is in the discrete series; see DeGeorge-Wallach [D-W]. On the other hand, (4) is way off when π is the trivial representation. What emerges from this work is the following simple conjecture which interpolates between these

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