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ON THE FIRST BETTI NUMBERS OF HYPERBOLIC SURFACES

XIAOXI XUE

Introduction. Let G be $SO(n, 1)$ or $SU(n, 1)$ and let K be a fixed maximal compact subgroup of G. Put on $B = G/K$ any G-invariant Riemannian structure. Let Γ be a cocompact, torsion-free, discrete subgroup of G and, for any irreducible unitary representation ω of G, let $N(\Gamma, \omega)$ denote the multiplicity of ω in $L^2(\Gamma\backslash G)$. Fix a Haar measure dG on G such that $\mathrm{vol}(\Gamma\backslash G) = \mathrm{vol}(\Gamma\backslash B)$.

One purpose of this paper is to give some estimates of $N(\Gamma, \pi_{1,-i\alpha\rho})$ where $\pi_{1,-i\alpha\rho}$ with $0 < \alpha \leqslant 1$ is the spherical complementary series in the standard notation. Let λ_α denote the eigenvalue of the Laplacian operator on a representative of $\pi_{1,-i\alpha\rho}$; then $i\alpha\rho = \sqrt{\lambda_\alpha - \rho^2}$.

The main tool is the Selberg trace formula. We first choose a suitable test function to get a technical lemma which is a more general form of R. Brooks's result [Br1] and which is much easier to obtain. Let $r = r(\Gamma)$ be the injectivity radius of the quotient space $\Gamma\backslash B$ and $B(r)$ be the ball in B with the radius r. Using our lemma, we prove the following theorem.

THEOREM. *There is a constant c which depends only on G such that, for any cocompact, torsion-free, discrete subgroup Γ with $r \geqslant 1/8$, if*

$$\frac{\mathrm{vol}(B(r))}{\mathrm{vol}(\Gamma\backslash B)^\sigma} \geqslant \eta, \qquad \sigma, \eta > 0,$$

then

$$\sum_{1 \geqslant \alpha \geqslant \alpha_0} N(\Gamma, \pi_{1,-i\alpha\rho}) \leqslant c\eta^{-\alpha_0} r \,\mathrm{vol}(\Gamma\backslash B)^{1-\alpha_0\sigma} \tag{1}$$

where $\alpha_0 > 0$.

THEOREM. *Given an arithmetic group Γ of G, there exists a constant c_Γ such that, for any congruence subgroup $\Gamma_{\mathscr{Q}}$ which is deep enough, we have*

$$\sum_{1 \geqslant \alpha \geqslant \alpha_0} N(\Gamma_{\mathscr{Q}}, \pi_{1,-i\alpha\rho}) \leqslant c_\Gamma S(\Gamma_{\mathscr{Q}}) \,\mathrm{vol}(\Gamma_{\mathscr{Q}}\backslash B)^{1-\sigma_0\alpha_0} \tag{2}$$

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