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ON COMPLETE QUATERNIONIC-KÄHLER MANIFOLDS

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1. Introduction. This article is concerned with the existence of complete Riemannian metrics of special holonomy on $\mathbb{R}^{4n}$. We therefore begin by recalling the basic notions and results concerning holonomy groups; cf. [5] [6] [23].

Let (M, g) be a connected Riemannian m-manifold, and let $x \in M$ be a chosen basepoint. The *holonomy group* of (M, g, x) is the subgroup of $\mathrm{End}(T_x M)$ consisting of those transformations induced by parallel transport around piecewise-smooth loops based at x; the *restricted holonomy group* is similarly defined, using only loops representing $1 \in \pi_1(M, x)$. The latter is automatically a connected Lie group and may be identified with a Lie subgroup of $\mathrm{SO}(m)$ by choosing an orthogonal frame for $T_x M$. Changing the basepoint and/or frame only changes this subgroup by conjugation.

Excluding Riemannian products and symmetric spaces, very few subgroups of $\mathrm{SO}(m)$ can be restricted holonomy groups, as was first pointed out by Berger [4]. In fact, the full list is as follows: $\mathrm{SO}(m)$, $\mathrm{U}(m/2)$, $\mathrm{SU}(m/2)$, $\mathrm{Sp}(m/4) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ $(m \geqslant 8)$, $G_2 (m = 7)$ and $\mathrm{Spin}(7)$ $(m = 8)$. In all but the first two cases, the manifold must be Einstein and must moreover be Ricci-flat except in the case of $\mathrm{Sp}(m/4) \times \mathrm{Sp}(1)/\mathbb{Z}_2$, for which the scalar curvature is *never* zero. A manifold of the latter holonomy group therefore resembles a symmetric space to an uncomfortable degree, and it behooves one to ask whether there are many or few complete manifolds of this type. In the positive scalar curvature case, there are no known complete nonsymmetric examples, and such are even known not to exist [21] in dimension 8; moreover, the moduli space of such metrics on a fixed manifold is a discrete space [15]; cf. [25]. In this article it will be shown that, by contrast, the moduli space of complete metrics on $\mathbb{R}^{4n}$ with holonomy $\mathrm{Sp}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ is infinite-dimensional. (The scalar curvature of these Einstein metrics is, of course, negative.)

A Riemannian manifold (M, g) of dimension $4n$, $n \geqslant 2$, will be called *quaternionic-Kähler* if its holonomy is (up to conjugacy) a subgroup of $\mathrm{Sp}(n)\mathrm{Sp}(1) := \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2$, but not a subgroup of $\mathrm{Sp}(n)$. Here $\mathrm{Sp}(n) := \mathrm{GL}(n, \mathbb{H}) \cap \mathrm{SO}(4n)$, where $\mathbb{H}$ denotes the quaternions, and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ is the subgroup of $\mathrm{SO}(4n)$ consisting of transformations of $\mathbb{R}^{4n} = \mathbb{H}^n$ of the form

$$\vec{v} \mapsto A\vec{v}\,q^{-1}$$

where $A \in \mathrm{Sp}(n)$ and $q \in S^3 \subset \mathbb{H}$. Such a manifold is never a Riemannian product

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