Vol. 63, No. 2 DUKE MATHEMATICAL JOURNAL © July 1991

BERNOULLI-EULER UPDOWN NUMBERS ASSOCIATED WITH FUNCTION SINGULARITIES, THEIR COMBINATORICS AND ARITHMETICS

V. I. ARNOLD

The numbers we shall study below are the numbers of topologically different maximal morsifications of critical points of smooth functions. In the simplest case of functions of one variable they are closely related to the Bernoulli and Euler numbers. But they are defined for any critical point of function. Since the classification of the critical points of functions is related to that of simple Lie algebras and to that of finite Euclidean Coxeter reflection groups, our construction associates the generalized Bernoulli-Euler numbers to the simple Lie algebras or to the Coxeter groups.

These numbers have remarkable combinatorial and arithmetical properties; some of these may be new even for the classical case.

1. Introduction. The *bifurcation diagram of functions* is the hypersurface in the space of functions, consisting of those functions which have either a degenerate critical point or two coinciding critical values.

The bifurcation diagram separates the components of its complement in the neighborhood of a given function. The representatives of these components will be called the morsifications of the given function.

A morsification (a component) is called maximal (for short, *M-morsification*), if the number of real nondegenerate critical points, born from the initial degenerate critical point, is maximal (equal to the multiplicity of the critical point).

Similar definitions can be applied to the critical points of real smooth functions of finite multiplicity (see [1], [2] for the details).

Example. Let us consider the morsifications of the function x^{n+1} of one variable. The M-morsifications are equivalent to those polynomials

$$x^{n+1} + a_1 x^{n-1} + \cdots + a_n$$

which have n different real critical values. The number K_n of nonequivalent M-morsifications is equal to the number of connected components, into which the bifurcation diagram cuts the above n-dimensional space of polynomials. The numbers K_n form the *updown sequence*

Received 3 May 1990. Revision received 19 November 1990.