Vol. 63, No. 2 DUKE MATHEMATICAL JOURNAL © July 1991

GEOMETRIC POSTULATION OF A SMOOTH FUNCTION AND THE NUMBER OF RATIONAL POINTS

J. PILA

1. Introduction. This paper is devoted to giving refinements and extensions of some of the results of Bombieri and the author [1] obtaining upper bounds for the number of integral lattice points on the graphs of functions. Consider a sufficiently smooth function $f(x)$ with graph Γ and a positive integer d. The main device of that paper was to consider integral points on Γ that do not lie on any real algebraic curve of degree d. The Main Lemma of [1] shows that such points cannot be too close together relative to certain norms of the function.

We pursue here two different goals relative to the Main Lemma. The first is to obtain local conditions on the function $f(x)$ that control the multiplicity of the intersection of Γ with any algebraic curve of degree d. This is essentially an investigation into the *hypotheses* of the Main Lemma and constitutes the "Geometric Postulation" of the title. Indeed, in Sections 2 and 3 we will obtain such conditions for any linear space of real algebraic curves.

The applications to integral points are given in Section 5. For example we show that, if $f(x) \in C^{104}$ on $[0, 1]$ and

$$W(f, 2) = f'' \begin{vmatrix} f''' & 3f'' & 0 \\ f^{iv} & 4f''' & 6f'' \\ f^{v} & 5f^{iv} & 20f''' \end{vmatrix}$$

is nowhere zero, then for every $\varepsilon > 0$

$$|t\Gamma \cap \mathbb{Z}^2| \leqslant c(f, \varepsilon)t^{1/2+\varepsilon}$$

where $t\Gamma$ is the homothetic dilation of Γ by a factor $t \geqslant 1$ (that is, the graph of $y = tf(x/t)$, $x \in t[0, 1]$). The same bound was proved in [1] under the assumption that f was C^∞ and strictly convex.

The second goal is an improvement of the Main Lemma itself that allows us to strengthen some of the results obtained therefrom. Thus, for example, let $f(x)$ be a transcendental analytic function on a closed bounded interval I, and Γ, the graph of f. It was shown in [1] that

$$|t\Gamma \cap \mathbb{Z}^2| \leqslant c(f, \varepsilon)t^{\varepsilon}$$

for every $\varepsilon > 0$. In particular, if $t = N$ is an integer, the same bound $c(f, \varepsilon)N^\varepsilon$ applies

Received 16 April 1990. Revision received 31 December 1990.