Vol. 63, No. 1 DUKE MATHEMATICAL JOURNAL © June 1991

RAGHUNATHAN'S TOPOLOGICAL CONJECTURE AND DISTRIBUTIONS OF UNIPOTENT FLOWS

MARINA RATNER

Introduction. In this paper we study distributions of orbits of unipotent actions on homogeneous spaces.

More specifically, let $\mathbf{G}$ be a real Lie group (all groups in this paper are assumed to be second-countable) with the Lie algebra $\mathfrak{G}$, Γ a discrete subgroup of $\mathbf{G}$ and $\pi: \mathbf{G} \to \Gamma\backslash\mathbf{G}$ the projection $\pi(\mathbf{g}) = \Gamma\mathbf{g}$, $\mathbf{g} \in \mathbf{G}$. The group $\mathbf{G}$ acts by right translations on $\Gamma\backslash\mathbf{G}$, $x \to x\mathbf{g}$, $x \in \Gamma\backslash\mathbf{G}$, $\mathbf{g} \in \mathbf{G}$.

Let $\mathbf{U}$ be a subgroup of $\mathbf{G}$ and $x \in \Gamma\backslash\mathbf{G}$. We say that the closure $\overline{x\mathbf{U}}$ of the orbit $x\mathbf{U}$ in $\Gamma\backslash\mathbf{G}$ is *homogeneous* if there is a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{U} \subset \mathbf{H}$, $\mathbf{xHx}^{-1} \cap \Gamma$ is a lattice in $\mathbf{xHx}^{-1}$, $\mathbf{x} \in \pi^{-1}\{x\}$, and $\overline{x\mathbf{U}} = x\mathbf{H}$. If these conditions are satisfied, we shall say that $\overline{x\mathbf{U}}$ is homogeneous with respect to $\mathbf{H}$.

Definition 1. A subgroup $\mathbf{U} \subset \mathbf{G}$ is called *topologically* rigid, if, given any lattice $\Gamma \subset \mathbf{G}$ and any $x \in \Gamma\backslash\mathbf{G}$, the closure of the orbit $x\mathbf{U}$ in $\Gamma\backslash\mathbf{G}$ is homogeneous.

A subgroup $\mathbf{U} \subset \mathbf{G}$ is called unipotent if for each $\mathbf{u} \in \mathbf{U}$ the map $\mathrm{Ad}_{\mathbf{u}}: \mathfrak{G} \to \mathfrak{G}$ is a unipotent automorphism of $\mathfrak{G}$.†

RAGHUNATHAN'S TOPOLOGICAL CONJECTURE. *Every unipotent subgroup of a connected Lie group* $\mathbf{G}$ *is topologically rigid.*

In fact, Raghunathan proposed a weaker version of this conjecture stated in [D1] and [M, Conjecture 2]. It was shown in [F1] and [P] that, if $\mathbf{G}$ is nilpotent, then every subgroup of $\mathbf{G}$ is topologically rigid. As to semisimple $\mathbf{G}$, it was shown in [H], [F2], and [DS] that for $\mathbf{G} = SL(2, R)$ the conjecture is true. Also, it was shown in [DM1] that certain unipotent subgroups of $SL(3, R)$ are topologically rigid. To the best of my knowledge these are the only cases of semisimple Lie groups for which the conjecture has been settled.

In this paper we prove the following

THEOREM A. *Every unipotent subgroup of a connected Lie group* $\mathbf{G}$ *is topologically rigid.*

Our method of the proof of Theorem A is totally different from that used in [DM1] for certain unipotent subgroups of $SL(3, R)$. We show that in order to prove Theorem A it suffices to prove it for one-parameter unipotent subgroups $\mathbf{U}$ of $\mathbf{G}$.

Received 30 July 1990. Revision received 25 February 1991.
Partially supported by the NSF Grant DMS-8701840.

† As was noted by Professor A. Borel, it would be more appropriate to use the term "Ad-unipotent" instead of "unipotent" to avoid confusion with unipotency in algebraic groups.