Vol. 62, No. 2 DUKE MATHEMATICAL JOURNAL © March 1991

CONVEX POLYTOPES, COXETER ORBIFOLDS AND TORUS ACTIONS

MICHAEL W. DAVIS AND TADEUSZ JANUSZKIEWICZ

0. Introduction. An n-dimensional convex polytope is *simple* if the number of codimension-one faces meeting at each vertex is n. In this paper we investigate certain group actions on manifolds, which have a simple convex polytope as orbit space. Let P^n denote such a simple polytope. We have two situations in mind.

(1) The group is Z_2^n, M^n is n-dimensional and $M^n/Z_2^n \simeq P^n$.

(2) The group is T^n, M^{2n} is $2n$-dimensional and $M^{2n}/T^n \simeq P^n$.

Up to an automorphism of the group, the action is required to be locally isomorphic to the standard representation of Z_2^n on $\mathbb{R}^n$ in the first case, or T^n on $\mathbb{C}^n$ in the second case. In the first case, we call M^{2n} a "small cover" of P^n; in the second, it is a "toric manifold" over P^n. First examples are provided by the natural actions of Z_2^n and T^n on $\mathbb{R}P^n$ and $\mathbb{C}P^n$, respectively. In both cases the orbit space is an n-simplex.

Associated to a small cover of P^n, there is a homomorphism $\lambda: Z_2^m \to Z_2^n$, where m is the number of codimension-one faces of P^n. The homomorphism λ specifies an isotropy subgroup for each codimension-one face. We call it a "characteristic function" of the small cover. Similarly, the characteristic function of a toric manifold over P^n is a map $\mathbb{Z}^m \to \mathbb{Z}^n$. A basic result is that small covers and toric manifolds over P^n are classified by their characteristic functions (see Propositions 1.7 and 1.8).

The algebraic topology of these manifolds is very beautiful. The calculation of their homology and cohomology groups is closely related to some well-known constructions in commutative algebra and the combinatorial theory of convex polytopes. We discuss some of these constructions below.

Let f_i denote the number of i-faces of P^n and let h_j denote the coefficient of t^{n-j} in $\sum f_i(t-1)^i$. Then $(f_0, \dots, f_n)$ is called the *f-vector* and $(h_0, \dots, h_n)$ the *h-vector* of P^n. The f-vector and the h-vector obviously determine one another. The Upper Bound Theorem, due to McMullen, asserts that the inequality $h_i \leqslant \binom{m-n+i-1}{i}$, holds for all n-dimensional convex polytopes with m faces of codimension one. In 1971 McMullen conjectured simple combinatorial conditions on a sequence $(h_0, \dots, h_n)$ of integers necessary and sufficient for it to be the h-vector of a simple convex polytope. The sufficiency of these conditions was proved by Billera and Lee and necessity by Stanley (see [Brønsted] for more details and references). Research on

Received 10 November 1989. Revision received 24 July 1990.

This paper was written at MSRI, Berkeley, where both authors were partially supported by NSF grant DMS 8505550 through MSRI. The first author was also partially supported by NSF grant DMS 8607968.