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K-THEORY AND PATCHING FOR CATEGORIES OF COMPLEXES

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Consider a pullback square of commutative noetherian rings and surjective homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & R_{12}. \end{array}$$

Our goal is to understand certain categories of complexes over R in terms of categories of complexes over the other three rings, and in particular to study the algebraic K-theory of these categories.

The classical and best understood example is the category of finitely generated projective R-modules, $\mathbf{P}(R)$. From Milnor [M], we know that

(1) the square

$$\begin{array}{ccc} \mathbf{P}(R) & \longrightarrow & \mathbf{P}(R_1) \\ \downarrow & & \downarrow \\ \mathbf{P}(R_2) & \longrightarrow & \mathbf{P}(R_{12}) \end{array}$$

is cartesian; i.e., the map from $\mathbf{P}(R)$ to the fiber product $\mathbf{P}(R_1) \times_{\mathbf{P}(R_{12})} \mathbf{P}(R_2) = \{(P_1, P_2, \alpha) | P_i \in \mathscr{P}(R_i),\ \alpha: P_1 \otimes_R R_2 \to R_1 \otimes_R P_2 \text{ an isomorphism}\}$ is an equivalence of categories; the inverse equivalence is given by pullback, or "patching".

(2) there is a boundary map $\partial: K_1(\mathbf{P}(R_{12})) \to K_0(\mathbf{P}(R))$ that is part of an exact Mayer-Vietoris sequence

$$K_1(R) \to K_1(R_1) \oplus K_1(R_2) \to K_1(R_{12}) \xrightarrow{\partial} K_0(R) \to K_0(R_1) \oplus K_0(R_2) \to K_0(R)$$

where $K_i(S)$ is defined to be $K_i(\mathbf{P}(S))$ for any ring S. The boundary map ∂ is constructed via patching in accordance with condition (1).

For applications in the theory of algebraic cycles (see, e.g., [RCAKT], [AC], [RCG]), one wants similar results for categories of modules having both finite

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