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GENERATORS FOR THE DEFINING IDEAL OF CERTAIN RATIONAL SURFACES

ANTHONY V. GERAMITA AND ALESSANDRO GIMIGLIANO

Introduction. Let $Z = \{P_1, \ldots, P_s\}$ be a set of s distinct points in $\mathbb{P}^2 = \mathbb{P}^2_K$ (K an algebraically closed field) and let $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s \subseteq R = K[w_1, w_2, w_3]$ be the defining ideal of Z. Let $\mathbb{P}^2(Z)$ be the surface obtained from $\mathbb{P}^2$ by blowing up the points of Z. In this paper we shall begin a detailed study of the defining ideals of certain projective embeddings of $\mathbb{P}^2(Z)$. We shall be particularly interested in the degrees of the elements in a minimal generating set for these ideals and, more generally, in the graded Betti numbers in a minimal free resolution of these ideals.

Such questions have been considered by many authors in the past. We mention specifically the work of Castelnuovo [Ca], Mumford [Mu] and, more recently, the deep study initiated by Green in [Gr] and continued in the work of Green-Lazarsfeld [G-L 1]. For the special varieties that we study our results are much stronger than the very general results obtained by the aforementioned authors.

In order to understand the projective embeddings of $\mathbb{P}^2(Z)$ we shall be considering, we recall some simple facts about ideals of points in $\mathbb{P}^2$.

We can write $I = \bigoplus_{d \geqslant \alpha} I_d$, where α is the least degree of a curve in $\mathbb{P}^2$ which contains Z. The Hilbert function of Z, denoted $H(Z, -)$ (or sometimes $H(R/I, -)$) is defined by:

$$H(Z, t) = \dim_k(R_t/I_t).$$

The general facts about $H(Z, -)$ that we shall use are well-known [see, e.g., D-G-M] and are best summarized by the following remarks about the first difference of $H(Z, -)$, i.e., in terms of $\Delta H(Z, t) = H(Z, t) - H(z, t-1)$:

(a) $\Delta H(Z, t) \geqslant 0$ for all t;
(b) $\Delta H(Z, t) \leqslant t + 1$ for all t, equality $\Leftrightarrow 0 \leqslant t \leqslant \alpha - 1$;
(c) $\Delta H(Z, t) \geqslant \Delta H(Z, t+1)$ for $t \geqslant \alpha$;
(d) $\sum_{i=0}^{\infty} \Delta H(Z, i) = s$.

Let σ be the least integer t for which $\Delta H(Z, t) = 0$. Then the ideal I can be generated by forms of degree $\leqslant \sigma$ [G-M].

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