Vol. 62, No. 1 DUKE MATHEMATICAL JOURNAL © February 1991

ENDPOINT BOUNDS FOR AN ANALYTIC FAMILY OF HILBERT TRANSFORMS

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1. Introduction. The role of curvature in Harmonic Analysis has received increasing attention in recent years. The point of departure for work in this area has been the connection between submanifolds of $\mathbb{R}^n$ and decay of the Fourier transform of compactly supported surface distributions. Such decay estimates fail for submanifolds contained completely in some hyperplane and in general the "amount" of curvature of the submanifold is related to the rate of decay of the Fourier transform of the distribution.

Well-known operators whose L^p boundedness is affected by curvature are singular integrals along submanifolds of $\mathbb{R}^n$. Consider, for example, the case of an operator given by convolution with a distribution which is singular along a submanifold of codimension 1. Certain distributions give rise to convolution operators which are bounded on some but not all L^p. If a distribution depends analytically on a parameter z, for a given z, what is the set of all p's for which the associated operator is bounded on L^p?

We study the case where the analytic family of distributions is obtained by taking $-z-1$ transverse derivatives of arclength measure on the parabola and doing so in a homogeneous way. For $1 < p \leqslant 2$, the operators H_z are easily seen to be unbounded on L^p when $\operatorname{Re} z < 1/p - 2$, and one can show using Calderón-Zygmund theory and interpolation that H_z are bounded on L^p when the above inequality is reversed. For the critical $z = 1/p - 2 + i\theta$, the kernel of H_z lacks the amount of smoothness required by the usual singular integral theory to establish L^p boundedness. Nevertheless, the curvature of the parabola makes up for this lack of smoothness and enables us to prove positive results when the usual methods do not apply. For $p = 1$ we prove that these operators map H^1 to weak L^1 and for $1 < p < 2$ that they map L^p to weak L^p. We also prove that the first result is sharp in the sense that for $p = 1$ all these operators, except one, do not map L^1 to weak L^1. Precise statements of results are given in Section 2.

2. Preliminaries and statements of results. We denote by C_0^∞ the set of smooth functions with compact support. Fix ψ an even function in $C_0^\infty(\mathbb{R})$ such that $\psi \geqslant 0$, $\psi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\psi \equiv 0$ off $[-1, 1]$. For $\operatorname{Re} z > -1$, define an analytic family of distributions D_z acting on test functions of the real variable u as follows:

$$\langle D_z, f\rangle = 2\Gamma\left(\frac{z+1}{2}\right)^{-1} \int |u-1|^z \psi(u-1) f(u)\, du.$$

Received 6 March 1989. Revision received 26 February 1990.