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THETA LIFTING FOR UNITARY REPRESENTATIONS WITH NONZERO COHOMOLOGY

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§1. Introduction. Let (G, G') be an irreducible type I reductive dual pair inside the symplectic group $Sp = Sp_{2n}(\mathbf{R})$ (see [12] for terminology). Let $\widetilde{Sp}$ be the metaplectic two-fold cover of Sp. For any subgroup E of Sp we let $\tilde{E}$ denote its inverse image in $\widetilde{Sp}$. In this paper we study local theta lifting between discrete series representations of $\tilde{G}'$ and unitary representations of $\tilde{G}$ with nonzero cohomology. It will be shown that if the "size" of G' is not greater than that of G and π' is a "sufficiently regular" discrete series representation of $\tilde{G}'$ then it has nonzero theta lift to $\tilde{G}$. The corresponding representation $\pi = \theta(\pi')$ is a unitary representation with nonzero cohomology. In this way by varying $\tilde{G}'$ we obtain a large collection of unitary representations of $\tilde{G}$ with nonzero cohomology and, for $G = SO(n, 1)$ or $SU(n, 1)$, all of them.

One significance of this correspondence is its application to global theta lifting, and thus (via Matsushima's formula) to the construction of nontrivial cohomology for locally symmetric spaces associated to discrete, cocompact subgroups of $\tilde{G}$. (The basis of such application lies in the fact that for the discrete series representation π' there will always be an appropriate congruence subgroup Γ of $\tilde{G}'$ such that π' occurs in the cuspidal spectrum of $L^2(\Gamma'\backslash\tilde{G}')$. See [7], [27], and [28].) Examples of such global liftings were previously given by Kazhdan [17] (from U(1) to SU$(n, 1)$), Borel-Wallach [5] (from U(1) to SU(p, q)) and Anderson [3] (with G' compact). The theta liftings of Kudla-Millson and those of Tong-Wang (see for example [20], [29]), although of a somewhat different nature, are closely related.

In a forthcoming paper, it will be shown that many of the $A_q(\lambda)$'s obtained here via local theta lifting will indeed occur in $L^2(\Gamma\backslash G)$ for an appropriate cocompact congruence subgroup Γ. This will provide new nonvanishing results for the cohomology of locally symmetric spaces associated to such Γ. In this regard we wish to remark that the way in which we prove the main results of this paper is probably more important than the results themselves.

An irreducible type I dual pair is constructed as follows (cf. [12]). Let D be one of the three division algebras $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ (the quaternion algebra) over $\mathbf{R}$, with standard involution $\#$. Thus $\#$ is trivial in the first case, and is the complex (resp. quaternionic) conjugation in the last two cases. Let V, V' be finite-dimensional vector spaces over D endowed with nondegenerate $\#$-sesquilinear forms $(\, , \,)$ and $(\, , \,)'$, one $\#$-hermitian, and the other $\#$-skew-hermitian. Let G, G' be the isometry groups of $(\, , \,)$ and $(\, , \,)'$ respectively. Then (G, G') is an irreducible dual pair inside $Sp =$

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