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WAVEFRONT PROPAGATION FOR REACTION-DIFFUSION SYSTEMS OF PDE

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1. Introduction. M. I. Freidlin ([9], [10]) has introduced probabilistic techniques to study propagation for systems of reaction-diffusion PDE. The motivating idea is that should a reaction-diffusion system possess only a single unstable and a single stable equilibrium, then the solution u of the system will presumably tend to "switch" for large times from near the former to near the latter state. A mathematical problem is then to describe this transition, ideally in terms of simpler quantities than those governing the full, detailed behavior of the entire system of PDE. More precisely, to study the reaction-diffusion problem for large times of order ε^{-1}, Freidlin suggests a ε^{-1} rescaling in the space and time variables so that our attention turns to the solutions u^ε of certain ε-dependent systems of PDE. We then hope to show that as $\varepsilon \to 0$, the functions u^ε converge in some region $G \subset \mathbb{R}^n \times [0, \infty)$ to the stable point, and in the opposite region $[\mathbb{R}^n \times [0, \infty)]\backslash G$ to the unstable point. We simultaneously hope to describe geometrically and analytically this set G, whose boundary we envision as a spreading wavefront separating regions with quite different limiting behavior.

This paper, which is an extension to systems of earlier work [6] on single equations, brings to bear purely PDE techniques to this problem, especially the theory of viscosity solutions on Hamilton-Jacobi equations, due to Crandall-Lions [3]. The connection with the foregoing discussion is that the region G alluded to above is the set where the solution J of a certain Hamilton-Jacobi equation is negative. Our procedure for understanding the limiting behavior of the solution of the reaction-diffusion system of PDE is thus, first of all, to build an appropriate Hamiltonian H out of the data given in the problem, second, to solve the resulting Hamilton-Jacobi equation for J, and last, to demonstrate the different limiting behavior of the solutions u^ε of the scaled system on the sets $\{J < 0\}$ and $\{J > 0\}$. We informally regard the Hamiltonian as controlling somehow the rate of instability of the unstable point. We are thus able to characterize the asymptotic behavior of the "complicated" reaction-diffusion system in terms of the "simple" Hamilton-Jacobi equation. This possibility, first identified by Freidlin [9] in rather different terms, is attractive, but of course requires for its implementation many structural assumptions on the nonlinearities, which we list below. It would of course be quite

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