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PRESCRIBING GAUSSIAN CURVATURE ON S^2

ZHENG-CHAO HAN

1. Introduction. In this paper, we will study the following problem of Nirenberg: what functions K can be the Gaussian curvature of a metric g on S^2 which is pointwise conformal to the standard metric g_0? Let $K(x)$ be a given smooth function on S^2 and $g = e^{2u}g_0$ be a conformal metric to g_0, then the Gaussian curvature K_g of g is given by

$$K_g = e^{-2u}(1 - \Delta_{g_0} u),$$

where Δ_{g_0} is the Laplacian with respect to g_0. Thus the problem is reduced to solving the nonlinear PDE

$$\Delta_{g_0} u = 1 - Ke^{2u} \tag{1}$$

for some u.

There are some necessary conditions on K for this problem to be solvable. First the Gauss-Bonnet formula

$$\int_{S^2} Ke^{2u}\, dA_{g_0} = 4\pi$$

says that K has to be positive somewhere, here dA_{g_0} is the area element of g_0. Kazdan and Warner [KW] found another condition: If u solves (1), then

$$\int_{S^2} \langle \nabla K, \nabla x_j \rangle e^{2u}\, dA_{g_0} = 0; \tag{2}$$

here the coordinate functions x_j on S^2, $j = 1, 2, 3$, are the first eigenfunctions of Δ_{g_0}: $\Delta_{g_0} x_j + 2x_j = 0$, $j = 1, 2, 3$. Therefore, for $K = 1 + \varepsilon x_j$, (1) does not have a solution for $\varepsilon \neq 0$.

Moser [M] first proved the existence of a solution of (1) under the condition that K be an even function on S^2 and positive somewhere. The proof in [M] was based on Moser's sharp form of the Sobolev-Trudinger inequality: *There exist universal constants* $\alpha, C > 0$ *such that for all* $u \in H^1(S^2)$ *with* $\int_{S^2} u\, dA_{g_0} = 0$, $\int_{S^2} |\nabla u|^2\, dA_{g_0} \leqslant 1$,

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