Vol. 61, No. 2 DUKE MATHEMATICAL JOURNAL © October 1990

A GEOMETRIC SETTING FOR THE QUANTUM DEFORMATION OF GL_n

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Introduction. The usual definition of quantized enveloping algebras (Drinfeld [2], Jimbo [3]) is in terms of generators and relations. A more natural definition for the + part of these algebras has been given by Ringel [8]; in this definition, the algebra has a basis in 1-1 correspondence with the representations of a certain quiver, and the multiplication is also defined in terms of the quiver.

In this paper we construct the entire algebra (not only the + part) assuming that we are in type A, using the geometry of relative positions of pairs of flags in infinite dimensional space.

Our approach is as follows. We consider the set $\mathscr{F}$ of n-step filtrations of a vector space over a finite field F_q such that each subquotient is of countable dimension and each member of the filtration is in a fixed commensurability class of subspaces. There is a certain group acting naturally on the set of pairs of such filtrations; the orbits may be interpreted as relative positions of pairs of filtrations and they are indexed by $n \times n$ matrices with integer entries, which are $\geqslant 0$ off diagonal. We want to define an algebra structure on the complex vector space with basis given by the orbits, using the well-known procedure in the usual definition of a Hecke algebra. Namely, one would like to take as structure constants the quantities $c_{\mathcal{O},\mathcal{O}',\mathcal{O}''}$ for three orbits $\mathcal{O}, \mathcal{O}', \mathcal{O}''$, where $c_{\mathcal{O},\mathcal{O}',\mathcal{O}''}$ is the number of all $f \in \mathscr{F}$ such that $(f_1, f) \in \mathcal{O}$, $(f, f_2) \in \mathcal{O}'$ for a fixed $(f_1, f_2) \in \mathcal{O}''$. Unfortunately, this number may be infinite.

We can resolve this difficulty by a limiting procedure: we first define an algebra $\mathbf{K}_d$ using the geometry of pairs of n-step filtrations on a d-dimensional vector space and then study how its structure constants behave when d increases by a multiple of n. In the limit we obtain an algebra $\mathbf{K}$ (without unit element) over a ring of Laurent polynomials in an indeterminate v. Taking certain infinite sums in $\mathbf{K}$ we obtain directly the quantized enveloping algebra corresponding to GL_n; we also obtain a new definition of the finite dimensional algebras considered in [6] in the case where v becomes a root of 1.

At the same time we construct an "intersection cohomology basis" (in the spirit of [7]) for the algebra $\mathbf{K}$. This basis is related to the standard basis by a matrix whose entries may involve polynomials with negative coefficients (in contrast to what happened in [7]).

The reader is referred to [1] for a more algebraic approach to the quantized enveloping algebra of type A.

Received 11 May 1990.
*Supported in part by National Science Foundation grants DMS 8702842, DMS 8803083.