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UNIQUE CONTINUATION AND REGULARITY AT THE BOUNDARY FOR HOLOMORPHIC FUNCTIONS

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§0. Introduction and main results. The function $f(z) = \exp(-1/z^{1/3})$ is holomorphic in the upper half-plane, smooth in its closure, and vanishes of infinite order at the origin. We shall show (Theorem 1) that if a function h has these properties but also maps an interval containing the origin into a nonsingular C^2 curve, then h must vanish identically. When the curve is real analytic this reduces to the Schwarz reflection principle. We state our first result for a vector valued function.

We recall that if M is a submanifold of $\mathbb{C}^n$ of class C^1, we say that M is *totally real* if $T_m \cap JT_m = \{0\}$ for all $m \in M$, where T_m is the tangent space of M at m and J is the usual multiplication by $\sqrt{-1}$.

We define the Lipschitz space $\Lambda_\alpha(\mathbb{R}^n)$, $\alpha > 0$, as in, e.g., Stein [12]. In particular, $f \in \Lambda_1(\mathbb{R}^n)$ if $f \in L^\infty(\mathbb{R}^n)$ and there is a constant A such that $\|f(x+t) + f(x-t) - 2f(x)\|_\infty \leqslant A|t|$. Similarly $f \in \Lambda_k(\mathbb{R}^n)$, k a positive integer greater than 1, if $\partial f/\partial x_j \in \Lambda_{k-1}(\mathbb{R}^n)$. For α nonintegral, $\Lambda_\alpha(\mathbb{R}^n)$ is the usual Hölder space. A similar definition can be given for $\Lambda_\alpha(F)$, where F is a closed set of $\mathbb{R}^n$ with sufficiently smooth boundary.

Our first theorem gives both regularity and unique continuation results.

THEOREM 1. *Let Ω be an open neighborhood of 0 in $\mathbb{C}$, $\Omega^+ = \Omega \cap \{w = s + it: t > 0\}$, and M' a totally real manifold of $\mathbb{C}^n$ of class C^k, $k \geqslant 2$, with $0 \in M'$. If $h: \overline{\Omega^+} \to \mathbb{C}^n$ is continuous and holomorphic in Ω^+ and maps $\overline{\Omega^+} \cap \mathbb{R}$ into M' then $h \in \Lambda_k(\overline{\Omega'^+})$ for every open Ω' relatively compact in Ω. Furthermore, if h vanishes of infinite order at the origin, i.e., $h(w) = O(|w|^N)$ for every N, then h vanishes identically in the connected component of the origin in $\overline{\Omega^+}$.*

If M is a totally real manifold of class C^2, $M \subset \mathbb{C}^p$ of real dimension p, $0 \in M$, then M is given locally (see Lemma 1.1) by

$$(0.1) \qquad M = \{w \in \mathbb{C}^p: \operatorname{Im} w = \varphi(\operatorname{Re} w)\}$$

for some $\varphi \in C^2(U)$, U a neighborhood of the origin in $\mathbb{R}^p$, φ real valued with $\varphi(0) = \varphi'(0) = 0$. *A wedge $\mathscr{W}$ of edge M* is then defined as a set of the form

$$(0.2) \qquad \mathscr{W} = \{w \in \mathcal{O}: \operatorname{Im} w - \varphi(\operatorname{Re} w) \in \Gamma\},$$

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