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HODGE THEORY WITH LOCAL COEFFICIENTS ON COMPACT VARIETIES

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In [A2] we announced a theorem on the existence of mixed Hodge structures on the cohomology of algebraic varieties with coefficients in certain local systems. In this paper, we treat the case of compact varieties; the general case will be treated elsewhere. More precisely, we prove the following: let X be a compact analytic space, which is bimeromorphic to a Kähler manifold, and L a local system of $\mathbb{R}$-vector spaces arising from an orthogonal representation of $\pi_1(X)$; then $H^i(X, L)$ carries a functorial mixed Hodge structure. If X has normal crossing singularities then we can give a fairly precise description of the Hodge filtration on $H^i(X, L) \otimes \mathbb{C}$ (due to Friedman [Fr] when $L = \mathbb{R}$). As an application, we show that if X is smooth, $f: X \to C$ a surjective holomorphic map to a smooth curve and $\mathscr{L}$ a line bundle in $\mathrm{Pic}^0(X)$, then the sheaves $R^if * \mathscr{L}$ are locally free.

In the second part of this paper, we explore the consequences of these results for the fundamental group. If X is an analytic space as above which is normal, then we show that $\pi_1(X)$ cannot admit certain types of amalgamated free product decompositions. Results of this kind have been obtained for Kähler manifolds by Johnson and Rees [JR] and Gromov [G], using completely different ideas. When X is a Kähler manifold, Green and Lazarsfeld [GL] introduced a subset $S^1(X)$ of $\mathrm{Pic}^0(X)$ consisting of those line bundles with nontrivial first cohomology. A natural question is: under what condition is $S^1(X)$ finite? We give a sufficient condition: $S^1(X)$ is finite whenever $\pi_1(X)'/\pi_1(X)''$ is finitely generated, where G' denotes the commutator subgroup of G. As a consequence, $\pi_1(X)'$ is not finitely generated whenever X maps onto a curve of genus two or more.

§1. Hodge Theory. We will use the following terminology:

1) A compact analytic space X is bimeromorphically Kähler if there exists a (holomorphic) bimeromorphic map $Y \to X$, with Y a Kähler manifold. We shall abbreviate this by saying that X is B-Kähler. Compact complex algebraic varieties and Moishezon spaces are examples. Varouchas [V, 3.2] has shown that the class of B-Kähler spaces coincides with Fujiki's class C [F]. (I am grateful to J. Kollár for this remark.)

2) A will denote $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. The term mixed Hodge structure, without any further qualification, will refer to one defined over A. A mixed Hodge structure H has weights in a set $S \subset \mathbb{Z}$, if $Gr_k{}^W H = 0$ for $k \notin S$.

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