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ON UNITARITY OF SPHERICAL REPRESENTATIONS

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0. Introduction. Let G be a semisimple Lie group with finite center and choose a maximal compact subgroup K in G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0 = \mathrm{Lie}(G)$. One of the main unsolved problems in the theory of semisimple Lie groups is to classify the set of unitary irreducible representations of G up to equivalence. Although a lot of progress has been made in recent years the methods used are still case by case analysis. The problem has two steps; namely, to exclude certain representations and to prove that the remaining representations are unitary. This work is aimed at the first step.

So let (π, H_π) be an irreducible representation of G. By a well-known result of Harish Chandra, H_π is equivalent to a unitary representation if and only if the corresponding $(\mathfrak{g}, K)$-module, X_π, admits a positive definite invariant Hermitian form, say ω. It is well known (in terms of the Langlands data) which X_π's admit an invariant Hermitian form. So to decide if (π, H_π) is unitary one "only" has to check if ω is positive definite on all K-types in X_π. The main obstacle here is of course that ω is only implicitly given except in a few low rank cases. Now very often ω is obtained by inducing an invariant Hermitian form on a spherical (i.e., a representation having a K-fixed vector) representation on a smaller (parabolic) subgroup. Hence it is important to understand the spherical representations. The work of Vogan [9] for $GL(n)$ and Barbasch [2] for G complex and classical suggests that one only has to know the signature of ω on a very small (on the order of $\mathrm{rank}(G)$) set of K-types, but it is not known which K-types to consider.

In this paper we define a small set of K-types $S(G)$. These are in some sense the smallest nontrivial K-types that can occur in a spherical representation. Also if (π, H_π) is a nontrivial spherical representation, then at least one K-type in $S(G)$ will occur in H_π. Moreover the set is closed under restriction to parabolic subgroups in the following sense: if $MN \subseteq G$ is the Levi decomposition of a parabolic subgroup and $\mu \in S(G)$, then $\mu|_{M \cap K}$ breaks up into a sum of the trivial $K \cap M$-type, a sum of $K \cap M$-types in $S(M)$ and a sum of nonspherical $M \cap K$-types. The main motivation for choosing this set of K-types is that if G is simple, classical and has split rank one, then $S(G)$ is just the irreducible subrepresentations of $\mathfrak{p}$ (Proposition 3.2) and an irreducible spherical representation with integral infinitesimal character admitting an invariant Hermitian form is unitarizable if and only if the form is positive on the μ-isotypic space for μ an irreducible subrepresentation of $\mathfrak{p}$. Moreover for G general and $J^G(\nu)$ (notation (1.10)) an irreducible spherical representation admitting an invariant Hermitian form such that $\|\nu\| \gg 0$, then $J^G(\nu)$ is not

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