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THE SPECTRAL GEOMETRY OF FLAT DISKS

ROBERT BROOKS, YAKOV ELIASHBERG AND C. MCMULLEN

In [6], Mark Kac raised the question of whether two domains in the Euclidean plane $\mathbb{R}^2$ which have the same spectrum of the Laplacian are congruent. Central to his approach to this question was the fact that certain invariants of the Laplacian for such a domain D, called the heat invariants, are expressible in terms of the area of D and integrals over the boundary ∂D of the domain of terms involving the length, the geodesic curvature, and derivatives of these quantities—see §1 below for a discussion.

One is naturally led by this approach to consider the extent to which the geometry of D in a neighborhood of ∂D governs the spectrum of D. To that end, let us say that two flat disks D_1 and D_2 are isometric near the boundary if there are neighborhoods of ∂D_1 and ∂D_2 which are isometric. Furthermore, we will say that D_1 and D_2 are piecewise isometric near the boundary if there are neighborhoods N_1 and N_2 of ∂D_1 and ∂D_2 such that N_1 can be subdivided into finitely many pieces and rearranged to obtain N_2.

It is easy to see that if D_1 and D_2 are piecewise isometric near the boundary and have the same area, then all of the heat invariants of D_1 and D_2, and indeed all such integrals over the boundary, must agree. It is also easy to see that if D_1 and D_2 are planar disks which are isometric near the boundary, then D_1 and D_2 are congruent.

However, it is an interesting fact that there are flat disks D_1 and D_2 which are isometric near the boundary, but which are not themselves isometric. Such disks immerse into the plane so that they have a common boundary curve. We will see how to construct such examples in §2 below.

In §3 and §4 below, we will then show:

THEOREM 1. *There are compact flat disks D_1 and D_2 which are isometric near the boundary, but which are not isospectral for either Dirichlet or Neumann boundary conditions.*

Using similar techniques, we will also show:

THEOREM 2. *There exist planar disks D_1 and D_2 which are piecewise isometric near the boundary, but which are not isospectral for either Dirichlet or Neumann boundary conditions.*

Finally, in §5 we will establish the analogue of Theorem 1 in dimensions greater than 2.

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