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HOMOTOPY CLASSES OF $*$-HOMOMORPHISMS BETWEEN STABLE C^*-ALGEBRAS AND THEIR MULTIPLIER ALGEBRAS

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0. Introduction. Let A and B be C^*-algebras and let $\mathscr{K}$ denote the compact operators on an infinite-dimensional separable Hilbert space. The homotopy classes of $*$-homomorphisms $A \to \mathscr{K} \otimes B$ form an abelian semigroup which we denote by $[A, \mathscr{K} \otimes B]$. In the first section we study the two-variable functor $N(A, B)$ obtained by applying the Grothendieck construction to $[A, \mathscr{K} \otimes B]$. It is shown that N is the universal functor from the category of C^*-algebras to the category of abelian groups which is stable, homotopy invariant and additive. This is in complete analogy with the KK-functor which was shown by Higson to be the universal such functor which is stable, homotopy invariant and split-exact.

In Section 2 we give two alternative descriptions of $N(A, B)$ in the case when A and B are σ-unital. To describe them, call a $*$-homomorphism $\phi: A \to \mathscr{K} \otimes B$ quasi-unital when there is a projection p in the multiplier algebra $\mathscr{M}(\mathscr{K} \otimes B)$ of $\mathscr{K} \otimes B$ such that $\overline{\phi(A)\mathscr{K} \otimes B} = p\mathscr{K} \otimes B$, and call ϕ unital when $p = 1$. The homotopy classes $[A, \mathscr{K} \otimes B]_1$ of unital $*$-homomorphisms form an abelian semigroup and it is shown that the group obtained by applying the Grothendieck construction to $[A, \mathscr{K} \otimes B]_1$ agrees with $N(A, B)$, when there is any unital $*$-homomorphism $A \to \mathscr{K} \otimes B$ at all. Practically the same proof of this fact also shows that the semigroup $[A, \mathscr{K} \otimes B]_q$ of homotopy classes of quasi-unital $*$-homomorphisms $A \to \mathscr{K} \otimes B$ is isomorphic to $[A, \mathscr{K} \otimes B]$.

Generally the calculation of $[A, \mathscr{K} \otimes B]$ and $N(A, B)$ is difficult even for abelian A and B. But when both A and B are $AF - C^*$-algebras the calculation is fairly simple and is carried out in Section 3. It turns out that $N(A, B)$ is isomorphic to the subgroup of $\mathrm{Hom}(K_0(A), K_0(B))$ generated by the positivity preserving elements.

In the final and longest section we apply the result of Section 2 to give a new description of the KK-groups and the Kasparov product. We adopt the point of view of J. Cuntz, that the Kasparov product is a generalization of the composition of $*$-homomorphisms but carry the construction of the product through with a definition of the KK-groups which is much closer to Kasparov's original than the definition of Cuntz based on quasi-homomorphisms. To be specific, we define a $kK(A, B)$-cycle to be a pair of $*$-homomorphisms $\phi_+, \phi_-: \mathscr{M}(\mathscr{K} \otimes A) \to \mathscr{M}(\mathscr{K} \otimes B)$ that are both continuous with respect to the strict topology of the two multiplier algebras and satisfy that $\phi_+(x) - \phi_-(x) \in \mathscr{K} \otimes B$, $x \in \mathscr{K} \otimes A$. After identifying kK-cycles that are homotopic we obtain the group $kK(A, B)$. This point of view is

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