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A CONVEXITY THEOREM FOR ISOSPECTRAL MANIFOLDS OF JACOBI MATRICES IN A COMPACT LIE ALGEBRA

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1. Introduction. We begin by presenting an example which will illustrate most of our results and some of the technique. Let K be the compact group $SU(l+1)$, and let $\mathscr{K}$ be its Lie algebra $su(l+1)$. Later on, $\mathscr{K}$ will be the compact form of an arbitrary complex semisimple Lie algebra. A *Jacobi matrix* is a (skew-hermitian) tri-diagonal matrix $L \in \mathscr{K}$

$$L = \begin{pmatrix} i\beta_1 & a_1 & 0 & \dots & 0 \\ -\bar{a}_1 & i\beta_2 & a_2 & \dots & 0 \\ 0 & -\bar{a}_2 & i\beta_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_l \\ 0 & 0 & \dots & -\bar{a}_l & i\beta_{l+1} \end{pmatrix} \tag{1}$$

(the a_j are complex, the β_j are real and satisfy $\Sigma\,\beta_j = 0$.) Let *Jac* be the set of Jacobi matrices. As a special case, we could choose the a_j to be pure imaginary. Up to a factor of $\sqrt{-1}$, the matrix L would then be real, symmetric, and tri-diagonal. An *isospectral manifold* is a subset of *Jac* consisting of Jacobi matrices with fixed eigenvalues $i\lambda_1, \dots, i\lambda_{l+1}$. All the skew-hermitian matrices with those eigenvalues are obtained from $\Lambda = \mathrm{diag}(i\lambda_1, \dots, i\lambda_{l+1})$ by conjugation; they lie on an orbit of the adjoint action of K on $\mathscr{K}$,

$$\mathcal{O}_\Lambda = \{k\Lambda k^{-1} \mid k \in K\}.$$

The Jacobi matrices conjugate to Λ are a small submanifold of $\mathcal{O}_\Lambda$. We require from now on that *all eigenvalues $i\lambda_j$ be distinct*. In that case, $\dim\mathcal{O}_\Lambda = l(l+1)$, while the set of Jacobi matrices conjugate to Λ has dimension $2l$.

For sake of exposition, we will assume that the matrix (1) has the form

$$i \times \begin{pmatrix} \beta_1 & a_1 & 0 & \dots & 0 \\ a_1 & \beta_2 & a_2 & \dots & 0 \\ 0 & a_2 & \beta_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_l \\ 0 & 0 & \dots & a_l & \beta_{l+1} \end{pmatrix}. \tag{2}$$

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