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THE LOWER BOUND IN THE LAW OF THE ITERATED LOGARITHM FOR HARMONIC FUNCTIONS

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§0. Introduction. Let $S_n = \Sigma_{k=1}^n X_k$, where $X_1, X_2, \ldots$ are independent random variables and let s_n denote the variance of S_n. Assume $s_n \to \infty$ as $n \to \infty$ and that

$$(K_0) \qquad |X_n|^2 \leqslant K_n \frac{s_n^2}{\log \log s_n^2} \quad a.s.$$

for some sequence of constants $K_n \to 0$ as $n \to \infty$. Then the celebrated law of the iterated logarithm (LIL) of Kolmogorov [10] states that for almost every ω,

$$(0.1) \qquad \limsup_{n\to\infty} \frac{S_n(\omega)}{\sqrt{2s_n^2 \log \log s_n}} = 1.$$

This beautiful result, whose roots lie in Borel's Theorem on normal numbers, was first proved by Khintchine [9] for Bernoulli random variables which trivially satisfy the Kolmogorov condition (K_0). Marcinkiewicz and Zygmund [12] proved that the condition (K_0) is best possible in the sense that if the sequence of constants K_n converging to 0 is replaced by a positive constant K then the conclusion fails. The LIL has been extended in several directions and we refer the reader to Bingham [3] for exhaustive (and exhausting) literature on this subject. In 1970, Stout [15] extended the LIL to martingales. Let $\{f_n\}$ be a martingale on a filtration $\{\mathscr{F}_n\}$ and let $\{d_n\}$ be its martingale difference sequence. Define the (conditional) square function by

$$\sigma_n^2 = \sum_{k=1}^n E(d_k^2 | \mathscr{F}_{k-1})$$

and if $n = \infty$, set $\sigma = \sigma_\infty$. Suppose the following condition holds:

$$(K_1) \qquad |d_n|^2 \leqslant K_n \frac{\sigma_n^2}{\log \log \sigma_n^2}$$

a.s. on $\{\sigma = \infty\}$ where K_n is a sequence of constants tending down to zero as $n \to \infty$.

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