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THE BERS EMBEDDING AND THE WEIL-PETERSSON METRIC

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Dedicated to Lipman Bers on the occasion of his seventy-fifth birthday

A special feature of the disc model for the hyperbolic plane is that the power series expansion of the metric $ds^2 = 4|dz|^2(1 + 2|z|^2 + 3|z|^4 + 4|z|^6 + \cdots)$ has vanishing pure-z terms and vanishing pure-$\bar{z}$ terms. More generally a local coordinate $z \in \mathbb{C}^n$ is k-normal for a Kähler metric g, provided the power series expansion in z of g has vanishing pure-z and pure-$\bar{z}$ terms for orders (total homogeneity) at most k, [G, Section 3.7]. A coordinate is ω-normal if it is k-normal for all positive k. For example the disc model provides an ω-normal coordinate for the hyperbolic metric, and the reader can check that the upper half plane model does not. Also note that given a Kähler metric g, and a tangent frame $\mathscr{F}$ at a point p then, provided it exists, the ω-normal coordinate z with $z(p) = 0$ and initial frame $\mathscr{F}$ is unique, [G].

Bers discovered a holomorphic embedding of Teichmüller space into the space of bounded holomorphic quadratic differentials, [B1, B2]. Our main result (see section 1.4 and Theorem 4.5) is quite simple, for the case of a compact Riemann surface the Bers embedding provides ω-normal coordinates for the Weil-Petersson metric. In fact we will see that the Bers embedding is uniquely characterized by this property.

Harmonic Beltrami differentials are at the center of our considerations. We find for z the deformation variable that all pure-z and pure-$\bar{z}$ derivatives for $z = 0$ of the hyperbolic area element (see Lemma 3.3), and the Weil-Petersson metric (see Theorem 4.5), vanish. These are instances of two general features of deformations by harmonic Beltrami differentials. The first: the derivatives of quantities defined on the individual Riemann surfaces and associated to the hyperbolic metric are vastly simplified for deformations by harmonic Beltrami differentials, see also [J2]. The second: for local coordinates coming from harmonic Beltrami differentials, a coordinate derivative, evaluated at the origin, of a quantity on Teichmüller space is in fact a Weil-Petersson covariant derivative, see also [W4].

And finally, both Professor Royden and the author have been aware of the main result for some time.

1. The Bers embedding and harmonic Beltrami differentials.

1.1. We briefly review the description of the Bers embedding of Teichmüller

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