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ON THE CONVEXITY OF A SOLUTION OF LIOUVILLE'S EQUATION

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1. Introduction. Let Ω be a simply connected domain in the complex plane $\mathbb{C}$, $\Omega \neq \mathbb{C}$, and let

$$g(z, \zeta) = -\log|z - \zeta| - h(z, \zeta)$$

be its Green's function (for the Laplacian operator). The function

$$h(z) = h(z, z)$$

plays an important role in geometric function theory (see e.g. [9], [16]) and in a number of physical applications ([2], [4], [6], [7], [8], [12], [15]).

As has been noticed in [7, p. 548], [8] (and is directly verified using (3) below), $h(z)$ satisfies Liouville's equation

$$\Delta h = 4e^{2h} \quad \text{in } \Omega. \tag{1}$$

The boundary behaviour is that $h(z) = -\log \delta(z) + O(1)$ as $z \to \partial\Omega$, where $\delta(z)$ denotes the distance to the boundary (see §3 below). Actually, $h(z)$ can be characterized as the unique solution of (1) with this boundary behaviour [3], [8], [15] and also as the maximum solution of (1) alone, [1, Lemma 1-1], [8], [15].

The main purpose of the present note is to give a simple, complex variable proof of the fact [3], [10], [11] that $h(z)$ is convex whenever Ω is convex. The major step in our proof can be formulated as a coefficient inequality for convex univalent functions. This part is not new [17], [5, Exercise 2, p. 70], but its connection with the convexity of $h(z)$ seems not to have been noticed in the literature. We also relate $h(z)$ to some other domain functions and expand a little on a physical interpretation of $h(z)$ in terms of vortex motion (thereby summarizing some parts of [8]).

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2. The main result. The following theorem was proved in [3]. Under suitable additional assumptions on $\partial\Omega$ it also becomes a special case of convexity results for solutions of more general PDEs (in arbitrary dimension) appearing in [10] and [11].

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