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BETTI NUMBERS OF HYPERSURFACES AND DEFECTS OF LINEAR SYSTEMS

ALEXANDRU DIMCA

0. Introduction. Let $\mathbf{w} = (w_0, \ldots, w_n)$ be a set of integer positive weights and denote by S the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ graded by the conditions $\deg(x_i) = w_i$ for $i = 0, \ldots, n$. For any graded object M, let M_k denote the homogeneous component of degree k. Let $f \in S_N$ be a weighted homogeneous polynomial of degree N with respect to $\mathbf{w}$.

Let V be the hypersurface defined by $f = 0$ in the weighted projective space $\mathbb{P}(\mathbf{w}) = \operatorname{Proj} S = \mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^*$ where the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ is defined by $t \cdot x = (t^{w_0}x_0, \ldots, t^{w_n}x_n)$ for $t \in \mathbb{C}^*$, $x \in \mathbb{C}^{n+1}$. Assume that the singular locus $\Sigma(f)$ of f is 1-dimensional, namely

$$\Sigma(f) = \{x \in \mathbb{C}^{n+1}; df(x) = 0\} = \{0\} \cup \left(\bigcup_{i=1}^{s} \mathbb{C}^* a_i\right)$$

for some points $a_i \in \mathbb{C}^{n+1}$, one in each irreducible component of $\Sigma(f)$.

Let G_i be the isotropy group of a_i with respect to the $\mathbb{C}^*$-action and let H_i be a small G_i-invariant transversal to the orbit $\mathbb{C}^* a_i$ at the point a_i. The isolated hypersurface singularity $(Y_i, a_i) = (H_i \cap f^{-1}(0), a_i)$ is called the **transversal singularity** of f along the branch $\overline{\mathbb{C}^* a_i}$ of the singular locus $\Sigma(f)$. Note that (Y_i, a_i) is in fact a G_i-invariant singularity.

The hypersurface V is a V-manifold (i.e., has only quotient singularities [8]) at all points, except at the points a_i where V has a **hyperquotient singularity** $(Y_i/G_i, a_i)$ in the sense of M. Reid [15].

In this paper we discuss an effective procedure to compute the Betti numbers $b_j(V) = \dim H^j(V)$ ($\mathbb{C}$ coefficients are used throughout) for such a weighted projective hypersurface V. It is known that only $b_{n-1}(V)$ and $b_n(V)$ are difficult to compute and that the Euler characteristic $\chi(V)$ can be computed (conjecturally in all, but surely in most of the interesting cases!) by a formula involving only the weights $\mathbf{w}$, the degree N and some local invariants of the G_i-singularities (Y_i, a_i) (see [6], Prop. 3.19). Hence it is enough to determinee $b_n(V)$.

On the other hand, it was known since the striking example of Zariski involving sextic curves in $\mathbb{P}^2$ having six cusps situated (or not) on a conic [25], that $b_n(V)$ is a very subtle invariant depending not only on the data listed above for $\chi(V)$ but also on the *position* of the singularities of V in $\mathbb{P}(\mathbf{w})$.

In the next three special cases the determination of $b_n(V)$ has led to beautiful and *mysterious* (see H. Clemens remark in the middle of p. 141 in [2]) relations with the

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