WHITTAKER MODULES ASSOCIATED
WITH HIGHEST WEIGHT MODULES

HISAYOSI MATUMOTO

§0. Introduction.   The material of this article is taken from my thesis at MIT. This article consists of two topics which are closely related. Namely, the one is the relation between the category of Whittaker modules and that of highest weight modules, the other is the relation between Whittaker functors and Casselman-Jacquet functors.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. We fix a Borel subalgebra of $\mathfrak{g}$ and denote by $\mathfrak{n}$ its nilradical. Let $\psi \colon \mathfrak{n} \to \mathbb{C}$ be a character and denote by Ψ the pair $(\mathfrak{n}, \psi)$. Let V be a left $U(\mathfrak{g})$-module. A vector $v \in V$ is called a (Ψ-)Whittaker vector if $Xv = \psi(X)v$ for all $X \in \mathfrak{n}$. A left $U(\mathfrak{g})$-module V is called a Whittaker module if V has a cyclic Whittaker vector. Whittaker modules are introduced by Kostant in [Ko3]. He clarified the structures of Whittaker modules and determined irreducible Whittaker modules when ψ is generic. In his thesis at MIT [Ly], Lynch generalized the notion of Whittaker modules. Namely, he generalized Kostant's theory to the characters on the nil-radicals of general parabolic algebras which have certain preferable properties. Such characters are called admissible (2.5). In this article, mainly, we assume admissibility of a character.

In [W2], Wallach generalized Kostant-Lynch's result. Mœglin [Mœ1,2,3] also studies Whittaker modules in a generalized situation and applies to the theory of Dixmier algebras (cf. [Vo2,4], [Mcg1,2]).

However, we cannot say Whittaker modules in the generalized sense are well-understood at this time. For example, in general, classifications of irreducible Whittaker modules are not known. One of important step of the study is to construct many examples of Whittaker modules systematically. In [Ko3], Kostant construct a Whittaker vector in the completion of a Verma module when $\mathfrak{n}$ is the nilradical of a Borel subalgebra. If we choose an irreducible generalized Verma module instead of a Verma module, his construction can extend to more general situation [Ly]. In [Ma3], Whittaker vectors in highest weight modules are studied. One of the purpose of this article is to study the module generated by such Whittaker vectors. Actually, we construct an exact functor $\Gamma_{\hat{\Psi}}$ from the category of highest weight modules to the category of Whittaker modules.

The other purpose of this article is as follows. Let G be a connected real semisimple linear Lie group whose complexified Lie algebra is $\mathfrak{g}$. In [GW], Goodman and Wallach proved the Whittaker vectors in Verma modules of Kostant can be regard