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ON THE QUASI-ISOMETRY INVARIANCE OF L^2 BETTI NUMBERS

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Introduction. In [4] and [5], the author defined "renormalized" L^2-Betti numbers for noncompact manifolds of bounded geometry equipped with a regular exhaustion. These Betti numbers depend on the germ of the spectrum of the Laplacian near zero, and not just on the space of harmonic forms. Thus their relation to L^2-cohomology is not obvious. However, in [5] an index theorem was proved relating the alternating sum of the Betti numbers to the "mean Euler characteristic," measured in terms of the regular exhaustion. It is easy to see that the mean Euler characteristic is an invariant of (strict) quasi-isometry (which may be considered as the analogue of diffeomorphism in the category of manifolds of bounded geometry), so it is natural to ask whether the Betti numbers taken individually are also invariants of quasi-isometry. The object of this paper is to answer that question in the affirmative. In [6], we apply this result to deduce a form of Morse theory for these noncompact manifolds.

What is in a sense the main idea of this paper is contained in §5. Our Betti numbers are defined as the limits of renormalized traces of certain approximate spectral projections of the Laplacian, and we show in that section by a sort of quantitative Hodge theory that if Q is such an approximate spectral projection (relative to one metric) then there is an approximate spectral projection P (relative to the other metric) such that P^2-PQP is small. We would like to deduce that the trace $\tau(P)$ is less than the trace $\tau(Q)$ up to some small error, and hence that the Betti numbers relative to the second metric are less than or equal to those relative to the first metric; a symmetry argument would then complete the proof. It is tempting to argue that $\tau(PQP) \leqslant \tau(Q)$, but this is false in general. (I was rescued from this error by C. Lazarov's careful reading of a preliminary draft of this paper.) However, if P and Q were projections in some type II_∞ von Neumann algebra, and if P-PQP were exactly zero, then $\tau(P) \leqslant \tau(Q)$ would follow from the comparison theory of projections (Takesaki [9] Chapter V). In general it is not clear whether the algebra of uniform operators can be extended to a von Neumann algebra on which τ is a normal trace; such an extension would give a more conceptual approach to §§3, 4 and 6 of this paper, in which we carry out certain algebraic constructions with "approximate" projections while remaining within the algebra of uniform operators. Making use of these constructions we are able in §7 to show that $\tau(P) \leqslant \tau(Q)$ up to a small error and so to complete the proof.

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