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ON THE (GENERALIZED) KORTEWEG-DE VRIES EQUATION

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§1. Introduction. In this paper we are concerned with the initial value problem (IVP) for the Korteweg-de Vries equation

$$(1.1)\qquad \begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & x, t \in \mathbb{R} \\ u(x, 0) = u_0(x) \end{cases}$$

and its generalized form

$$(1.2)\qquad \begin{cases} \partial_t u + \partial_x^3 u + a(u)\partial_x u = 0 & x, t \in \mathbb{R} \\ u(x, 0) = u_0(x) \end{cases}$$

where a is a function on $\mathbb{R}$ to $\mathbb{R}$ with $a(0) = 0$ and regularity to be specified later.

We shall study the well-posedness of these problems in the classical Sobolev spaces $H^s(\mathbb{R})$, and the regularity of their solutions in the spaces $L_s^p = (1 - \Delta)^{-s/2}L^p(\mathbb{R})$. In well-posedness we include existence, uniqueness, persistence property (i.e., the solution $u(t)$ at the time $t \in [-T, T]$ belongs to the same function space X as does the initial data u_0, and describes a continuous curve in X), and the continuity of the map $u_0 \to u(t)$ from X to $C([-T, T]: X)$. If $T = T(\|u_0\|_X) < \infty$ we call it local well-posed in X. In the case when T can be taken arbitrarily large the problem is globally well-posed in X.

One of our main results below is the proof of a global (in space) smoothing effect for solutions of these equations. To explain it we consider first the associated linear problem (i.e., $a(\cdot) \equiv 0$ in (1.2)) with $u_0 \in L^2(\mathbb{R})$. In this case the solutions $u(t)$ is given by the unitary group $\{W(t)\}_{-\infty}^{\infty}$. Thus, $u(t) = e^{it\partial_x^3}u_0 = W(t)u_0 \in C(\mathbb{R}: L^2(\mathbb{R}))$. From the results in [12] [13] [19] one has the following R. S. Strichartz type of result [17]:

$$(1.3)\qquad \left(\int_{-\infty}^{\infty} \|W(t)u_0\|_p^q \, dt\right)^{1/q} \leqslant c\|u_0\|_2$$

with $2/q = 1/3 - 2/3p$ and $p \in [2, \infty]$.

On the other hand, in [8] T. Kato has shown that solutions of the IVP (1.2) possess a local smoothing effect. In the linear case and when $u_0 \in L^2(\mathbb{R})$ this can be

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