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ON ARTIN'S CONJECTURE AND THE CLASS NUMBER OF CERTAIN *CM* FIELDS, I

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0. Introduction. Recent work of Gross and Zagier [3] contains the important result (built on work of Goldfeld [2]) that for K an imaginary quadratic field with discriminant $-D_K$, and $\varepsilon > 0$, $h_K \gg (\log D_K)^{1-\varepsilon}$, where h_K is the class number of K and the implied constant is effective. This is intimately connected with the Riemann hypothesis for $\zeta_K(s)$, the zeta function of K, for it is equivalent to showing that if β is a real zero of $\zeta_K(s)$, then $1 - \beta \gg (\log D_K)^{1-\varepsilon}/\sqrt{D_K}$, with an effective implied constant. Ultimately, one would like to show that β does not exist, or at least that a bound such as $1 - \beta > (3 \log D_K)^{-1}$ is true, for it would follow from this that $h_K > \sqrt{D_K}/(8 \log D_K)$.

The problem of finding effective lower bounds for class numbers of imaginary quadratic fields can be generalized to the situation where K is a totally complex number field containing a totally real field k, with $[K:k] = 2$ and $[k:\mathbb{Q}] = n$. Letting D_M denote the absolute value of the discriminant of a number field M, we have here $D_K = D_k^2 f$, where f is the norm of the relative different. In addition to asking how h_K grows as $f \to \infty$, which is analogous to the case where $k = \mathbb{Q}$ and K is imaginary quadratic, one can also consider the growth of h_K as $D_k \to \infty$ for fixed n, and as $n \to \infty$. The zeta function of K factors into the zeta function of k multiplied by a real Ray class L-series: $\zeta_K(s) = \zeta_k(s)L(s, \chi)$, where f is the norm of the conductor of χ, and in analogy to the case of the rational groundfield, the solution to the class number problem hinges upon the existence of a sufficiently large zero-free region about 1 for $L(s, \chi)$.

Let $\mathcal{N}$ denote the set of all totally real fields k such that a sequence of fields $\mathbb{Q} = k_0 \subset k_1 \subset \cdots \subset k_l = k$ exists, with each field normal over the preceding field. In [11] Stark showed that for $k \in \mathcal{N}$, under certain conditions a simple real zero β of $L(s, \chi)$ must also be a zero of the zeta function of an imaginary quadratic field contained in K. This implied that $1 - \beta$ must be bounded below by $D_K^{-1/2n}$. Combining this with the well-known fact that if β is a multiple zero of $\zeta_K(s)$, then $1 - \beta$ is bounded below by $(\log D_K)^{-1}$, he obtained $1 - \beta \gg \min(D_k^{-1/n} f^{-1/2n}, (\log D_k^2 f)^{-1})$, where the implied constant is effective and independent of k, K, f, n.

This was sufficiently strong to show that for $k \in \mathcal{N}$, $h_K \to \infty$ as $f \to \infty$ or as $D_k \to \infty$, n fixed, but did not quite suffice to show $h_K \to \infty$ as $n \to \infty$. This was due to a lack of sharp estimates for $\zeta_K(\sigma)$, $\sigma > 1$, when D_k is small. These difficulties were

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