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ON KOLLÁR'S VANISHING THEOREMS, I

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0. Introduction. Recall the classical vanishing theorem:

(0.0) THEOREM. *Let X be a projective and nonsingular variety over* $\mathbf{C}$ *and* $\mathscr{L}$ *an invertible ample sheaf on* X. *Then* $H^q(X, \mathscr{L} \otimes \Omega_X^p) = 0$ *whenever* $p + q > \dim X = n$ ■

This theorem was proved for $p = n$ by Kodaira in a celebrated paper [18]. The general case was done by Nakano [22] and had been observed independently by Serre. See also [21]. In a remarkable work, Kollár has recently extended Kodaira's theorem in a new direction [20]:

(0.1) THEOREM. *Let X be a projective nonsingular algebraic variety over* $\mathbf{C}$ *and* $\pi: X \to Y$ *a surjective morphism to a reduced variety Y. Then*

(i) *for all $i \geqslant 0$, the sheaves $R^i\pi_*\omega_X$ are torsion-free on Y (here $\omega_X = \Omega_X^n$ is the dualizing sheaf of top-order differential forms), and*

(ii) *if $\mathscr{L}$ is an ample invertible sheaf on Y, then, for all $i \geqslant 0$, $H^q(Y, \mathscr{L} \otimes R^i\pi_*\omega_X) = 0$ whenever $q > 0$.* ■

When $X = Y$ and $\pi =$ identity, statement (ii) above is none other than Kodaira's theorem. We ask, is there a vanishing theorem of Kollár's type that would simultaneously generalize (0.1.ii) *and* (0.0)? In other words, is there a vanishing theorem related to the direct images of the Ω_X^p with $p < n$? The purpose of this paper is to establish such a theorem. In this first part, we do so only under the highly restrictive conditions that both X and Y *and* the morphism π are smooth.

Our vanishing theorem (1.4) is a statement concerning certain *complexes* of sheaves on Y. These complexes arise in the construction by Deligne and Zucker of Hodge structures on the Leray factors $H^s(Y, R^t\pi_*\mathbf{C})$, whose construction was extended by Zucker to the case of degenerating coefficients over curves in [26]. We formulate our theorem for an arbitrary variation of Hodge structures over Y. From this viewpoint, our result can be considered a supplement to the investigations of Griffiths concerning global properties of variations of Hodge structures [11]. That Hodge theory should play a central role in vanishing theorems is no surprise. Indeed, the failure of the vanishing theorem to hold in characteristic $p > 0$ stems from the failure of Hodge theory to hold (in naive form) in positive characteristic. Moreover, the essential equivalence *Lefschetz hyperplane theorem* $\Leftrightarrow$ *classical vanishing theorem*, which holds modulo Hodge theory was known for some time (see [19]) but was exploited fully by Ramanujam [24] in his proof of the vanishing theorem.

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