Vol. 59, No. 2 DUKE MATHEMATICAL JOURNAL © October 1989

NON-KÄHLER MANIFOLDS AND BALL-PRODUCT CUSP SINGULARITIES

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Dedicated to Professor Ichiro Satake for his sixtieth birthday

In the present paper we construct compact complex manifolds of dimension $r(s+1)$ for $r \geqslant 2$, $s \geqslant 1$, possessing no Kähler metrics by using the method of toroidal embeddings [KKMS] and [AMRT] such that they are desingularizations of certain normal isolated singularities, which we shall define in section 1 and call "ball-product cusp" singularities.

Inoue [I] constructed a new compact complex surface of class VII_0, called Inoue-Hirzebruch surface, by gluing two desingularizations of appropriate Hilbert modular cusp singularities. Along the same line, Sankaran [Sn1] generalized Inoue-Hirzebruch surfaces to higher-dimensional non-Kähler manifolds by gluing two desingularizations of Hilbert modular cusp singularities of higher dimension. Their fundamental groups are free abelian groups of rank equal to the dimension of the manifolds minus one. And in [Sn_2] Sankaran constructs examples of non-Kähler manifolds whose fundamental groups are free abelian with rank less than the dimension of the manifolds.

Along another line, Tsuchihashi [T] constructed compact complex non-Kähler manifolds, called Inoue-Kato manifolds, with infinite cyclic fundamental groups. In the two-dimensional case, they are Inoue-Hirzebruch surfaces or half-Inoue surfaces, and in the three-dimensional case one of them is bimeromorphic to that constructed by Kato [K].

Our construction is analogous to those of [I] and [Sn1]. We glue a desingularization of a ball-product cusp singularity and something which is, say, a quotient of the product of a ball and an open set outside a ball. We do not know whether it can be contracted to a subvariety of lower dimension or not. For integers $r \geqslant 2$ and $s \geqslant 1$ we construct in section 2 a compact complex non-Kähler manifold of dimension $r(s+1)$. Its fundamental group is the semidirect product of free abelian groups of rank $2rs$ and $r-1$. And it has the first Betti number equal to $r-1$ but has no global holomorphic 1-forms. In section 3 we construct a one-parameter degeneration of our manifold X and show that the arithmetic genus $\chi(\mathcal{O}_X)$ and the index $\tau(X)$ of the manifold vanish. In section 4 we construct a family of deformations of X which is simultaneously contracted to a versal family of deformations of a ball-product cusp singularity.

We use in this paper the notations from [OM] or [Od] for torus embeddings.

Received August 10, 1988. Revision received December 12, 1988.