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THE NUMBER OF INTEGRAL POINTS ON ARCS AND OVALS

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1. Introduction. In 1926, Jarnik [4] proved that a strictly convex arc $y = f(x)$ of length ℓ contains at most

$$3(4\pi)^{-1/3}\ell^{2/3} + O(\ell^{1/3})$$

integral lattice points, and that the exponent and constant are best possible.

However, Swinnerton–Dyer [10] showed that the preceding result can be substantially improved if we start with a fixed, C^3, strictly convex arc Γ and consider the number of lattice points on $t\Gamma$, the dilation of Γ by a factor t, $t \geqslant 1$. This of course is the same as asking for rational points $(m/N, n/N)$ on Γ as $N \to \infty$. In fact, Swinnerton–Dyer proves a bound of type

$$|t\Gamma \cap \mathbb{Z}^2| \leqslant c(\Gamma, \varepsilon)t^{3/5+\varepsilon}$$

for $\varepsilon > 0$.

A little later, W. M. Schmidt [8] gave a uniform version of Swinnerton–Dyer's Theorem (with respect to Γ) and generalized it to higher dimensions. Schmidt proved that if $f \in C^3([0, N])$ with $|f| \leqslant N$ and $f''' \neq 0$ in $[0, N]$, then the number of integral points on the curve Γ: $y = f(x)$ does not exceed $c(\varepsilon)N^{3/5+\varepsilon}$ for every $\varepsilon > 0$, for some $c(\varepsilon)$ independent of f, and conjectured the result with exponent $\frac{1}{2}$. His result and conjecture are actually more precise, but we have stated them in a modified form for the sake of simplicity.

In this paper, we obtain a result which may be considered a first step toward Schmidt's conjecture, namely, that the hypotheses $f \in C^D([0, N])$, $|f| \leqslant N$, $|f'| \leqslant 1$, $f^{(D)} \neq 0$ in $[0, N]$ imply

$$|\Gamma \cap \mathbb{Z}^2| \leqslant c(\varepsilon_D)N^{1/2+\varepsilon_D}$$

where $\varepsilon_D \to 0$ as $D \to \infty$. We prove also an independent conjecture of Sarnak [7] that if $f \in C^\infty([0, 1])$ is strictly convex then

$$|t\Gamma \cap \mathbb{Z}^2| \leqslant c(f, \varepsilon)t^{1/2+\varepsilon}$$

for every $\varepsilon > 0$. In view of the example $f(x) = \sqrt{x}$, the exponent $\frac{1}{2}$ is best

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